Hanging nodes and XFEM

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SUMMARY

This paper investigates two approaches for the handling of hanging nodes in the framework of extended finite element methods (XFEM). Allowing for hanging nodes, locally refined meshes may be easily generated to improve the resolution of general, i.e. model-independent, steep gradients in the problem under consideration. Hence, a combination of these meshes with the XFEM facilitates an appropriate modeling of jumps and kinks within elements, that interact with steep gradients. Examples for such an interaction are, e.g., found in stress fields near crack fronts or in boundary layers near internal interfaces between two fluids. The two approaches for the XFEM based on locally refined meshes with hanging nodes basically differ in whether (enriched) degrees of freedom are associated with the hanging nodes or not. Both approaches are applied to problems in linear elasticity and incompressible flows. Copyright © 2000 John Wiley & Sons, Ltd.

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1. Introduction

The extended finite element method (XFEM) allows for the mesh-independent approximation of non-smooth solutions, cf. Moës and Belytschko [9, 40] and the recent overview provided by Fries and Belytschko [29]. Typical applications of the XFEM involve discontinuities, i.e. jumps or kinks, in the solutions. In addition, steep gradients are often encountered in the vicinity of discontinuities: Examples are found in stress fields near crack fronts and in boundary layers near internal interfaces between two fluids. In some cases, the enrichments used in the XFEM enable an appropriate treatment of both, discontinuities and additional steep gradients, which...
is for example the case for linear elastic fracture mechanics. There, the step-enrichment and the branch enrichment are applied along crack surfaces and at crack fronts, respectively. However, the enrichment functions for steep gradients are often model-dependent. For example, different enrichments at the crack front have to be used in cohesive and cohesionless crack models, cf. [37, 38, 59]. This is an important difference to the model-independent enrichments chosen for jumps and kinks. It is thus often desirable to use enrichments only for jumps and kinks and to resolve steep gradients by local mesh refinements. Furthermore, local mesh refinements are able to increase the resolution of interfaces where jumps and kinks occur. It is noted that the enrichment scheme for steep gradients as proposed in Abbas et al. [1] is truly model-independent. The idea in that scheme is to systematically span the complete range of gradients starting from those which cannot be captured by the standard finite element solution up to a true discontinuity. Another approach towards a model-independent enrichment scheme is proposed in Waisman and Belytschko [58], where an adaptive adjustment of the enrichments at the crack front is suggested.

Local mesh refinements with hanging nodes are particularly easy to realize compared to conforming refinements. Hanging nodes are those which exist not only in element corners but also on element edges/faces of their neighboring elements. Without loss of generality, we restrict ourselves to two-dimensional domains and accept hanging nodes only in the center of element edges ("2-to-1" property, 1-irregular mesh [6]). The refinement criteria employed herein are heuristical—in contrast to using error estimators—and based on physical considerations: we refine a priori near interfaces (including crack-tips) where steep gradients are expected or where the resolution is to be increased. The focus of this work is on these meshes with hanging nodes in the context of the XFEM. In particular, we also discuss the situation where hanging nodes are enriched, which is in contrast to Unger et al. [57] where the enrichment is only considered in standard finite elements without hanging nodes. For the success, it is crucial that

i. a conforming set of shape functions is found on the mesh with hanging nodes and
ii. these shape functions build a partition of unity.

We distinguish between two approaches depending on whether degrees of freedom (DOFs) are associated with the hanging nodes or not. The latter is often referred to as constrained approximation, cf. Ainsworth et al. [4, 3, 2], Solín et al. [48, 47], and Demkowicz et al. [23, 24].

In the first approach, where DOFs are present at hanging nodes, valid shape functions for all regular and hanging nodes have to be found so that (i) conformity and (ii) the partition of unity property are achieved. We use the shape functions of Gupta [32]. It is then found that the XFEM may be applied in a straightforward manner: The same enrichment functions and the same set of enriched nodes are used as in the standard case of the XFEM on conforming meshes. Special care is needed for the quadrature in this approach as—already without the enrichment—the special element shape functions have kinks in the reference element. An extension of these shape functions to three dimensions is found in Morton et al. [41]. Other approaches for obtaining conforming shape functions for the hanging nodes are e.g. proposed in Baitsch and Hartmann [5] and Cho et al. [20, 19]. To the best of our knowledge, the XFEM has so far not been used in the context of this approach where DOFs are present at the hanging nodes.

In the second approach, the fields at the hanging nodes are constrained to be the average of the neighboring corner nodes of the hanging node. With this approach, one may adopt two
different strategies that lead to identical results: In a first step, one employs the non-conforming bi-linear shape functions at the nodes and assembles the system matrix in the usual way for both strategies. In a second step, the constraints at the hanging nodes are (a) either enforced explicitly through Lagrange multipliers or (b) the system of equations is reduced through a multiplication with a so-called connectivity matrix. Thereby, the conformity (i) and partition of unity property (ii) are ensured.

In the latter case (b), the multiplication removes DOFs associated with hanging nodes from the system matrix and attributes their contributions to the DOFs of the neighboring (regular) nodes. In the XFEM, where standard finite element and enriched DOFs may be present at hanging nodes, it is important to attribute both types of DOFs to the DOFs of the neighboring (regular) nodes, respectively. Consequently, if a hanging node is to be enriched, then also the DOFs associated with the neighboring nodes have to be enriched. This constrained approximation has been previously used in the context of hp-adaptive XFEM, cf. Byfut et al. [16, 14, 15].

As different as both approaches—hanging nodes with and without DOFs—may seem, they still yield very similar results. Differences between the approaches may be found in the definition/construction of shape functions adjacent to hanging nodes and the resulting number of DOFs. Also, the sets of enriched nodes differ, as in constrained approximation, both neighboring nodes of a hanging node are enriched, if the hanging node itself is enriched. Furthermore, it is mentioned that the quadrature is simpler for the constrained approximation where it is identical to the standard XFEM.

This paper is organized as follows: In Section 2, the mesh refinement is described that leads to meshes with hanging nodes. Requirements on 1-irregular meshes only allowing for hanging nodes in the center of element edges are given in this section. Extended approximations as used in the XFEM are defined in Section 3 with enrichment functions for jumps and kinks. Section 4 describes the approach that uses DOFs at hanging nodes in detail. In particular, the definition of the shape functions, the coupling to the XFEM, and special issues such as quadrature are covered. Constrained approximation, where no DOFs at the hanging nodes exist, is described in Section 5. The different viewpoints for this approach are discussed and the extension to XFEM is explained. Numerical results are given in Section 6. The generality of the approaches discussed herein is stressed by considering applications from solid and fluid mechanics. Excellent results are achieved, proving that the XFEM and hanging nodes are a very promising combination. Section 7 concludes this paper.

2. Mesh refinement with hanging nodes

In this work, discontinuities are captured by means of the XFEM, however, steep gradients are resolved by local mesh refinements, see Figure 1. We only consider two-dimensional domains and the discretization is based on bi-linear elements. This restriction is only chosen for brevity and does not reduce the generality of the approaches discussed herein.

General mesh refinements may either maintain the conformity of the mesh or lead to irregular, i.e. non-conforming, meshes. Irregular meshes involve “hanging nodes” that are also referred to as “irregular vertices”. The differences between these two approaches are mainly found in the complexity of the refinement algorithm and the need for changes in the simulation codes that process the refined meshes. It is found that conforming mesh refinements are more
involved than those allowing for hanging nodes, in particular for three dimensions, cf. [4, 42]. However, the ability to account for hanging nodes needs to be incorporated into the simulation codes, whereas conforming refinements may be processed in standard finite element solvers. It is our belief that the ease of the refinement algorithm leading to irregular meshes outweighs the disadvantage of the need for adjustments in the simulation code. For conforming mesh refinements, we refer the interested reader to Niekamp and Stein [42], and for irregular mesh refinements to Ainsworth and Senior [4] and Bank et al. [6].

For irregular meshes, one may specify the irregularity index $k$ which denotes the maximum difference of refinement levels between adjacent elements in the mesh [6]. The meshes used in this work are (symmetric) 1-irregular. That is, for bi-linear elements, only one hanging node is accepted in the center of an element edge. This is also called the 2-to-1 condition/property in the literature. A graphical representation of these irregular meshes is given in Figure 2. Compared to meshes that employ no restriction on the index $k$, i.e. which allow for an arbitrary number of irregular nodes per edge, we find the following properties for 1-irregular meshes:

- The bandwidth of the resulting system of equations is bounded which does not necessarily hold for any $k \geq 2$ [6].
- The mesh size varies gradually in the domain, because the maximum factor of neighboring edge lengths is 2.
- The treatment of the hanging nodes in the simulation code is simplified as the local connectivity matrices have an identical structure compared to the situation for arbitrary numbers of hanging nodes per edge [4].

In particular in the context of $hp$-adaptivity driven by error estimators, it may still be preferable to allow for $k$-irregular meshes with $k > 1$. The XFEM-enrichment concepts stated in the following sections may similarly be applied to $k$-irregular meshes. Studies on the concept stated in Section 5 for higher-order XFEM on (unsymmetric) $k$-irregular meshes are in progress.

2.1. Definition of the 1-irregular mesh

Let $\Omega$ be a bounded, polygonal area of $\mathbb{R}^2$. Starting point is a conforming (coarse) mesh that discretizes the domain by $n^0_{el}$ shape regular quadrilaterals/elements. The collection of these
elements is called \( P_0 = \{1, \ldots, n_0^{el} \} \). The element area of element \( k \in P_0 \) is denoted by \( \omega_k^0 \subseteq \Omega \).

Furthermore, let there hold: (i) the domain is the union of all elements, \( \Omega = \cup_{k \in P_0} \omega_k^0 \), and (ii) the intersection of two distinct elements, \( \omega_i^0 \cap \omega_j^0 \), \( i, j \in P_0 \), is either empty, a vertex, or a shared edge.

Assume that based on this mesh, \( n_{ref} \) refinement steps are desired. The result of each refinement step is the collection of shape regular elements \( P_l \), \( 1 \leq l \leq n_{ref} \), with related element areas \( \omega_k^l \subseteq \Omega \), \( k \in P_l \), that fulfill the following properties: (i) \( \Omega = \cup_{k \in P_l} \omega_k^l \), (ii) \( \omega_i^l \cap \omega_j^l \), \( i, j \in P_l \), \( i \neq j \), is either empty, a vertex, a shared edge, or part of a shared edge, (iii) all vertices are either shared by other element vertices and/or in the center of element edges.

In contrast to the requirements on \( P_0 \), these requirements are in particular met by 1-irregular meshes.

Algorithmically, each collection \( P_l \) is obtained by applying the following procedure: (i) a set of elements \( W_{ref}^{l-1} \subseteq P_{l-1} \) is constructed which marks elements for (another) refinement, (ii) each element in \( W_{ref}^{l-1} \) is refined by sub-dividing it into four “children” elements (isotropic refinement) with nodes at the centroid and at the edge centers (the original element is called “parent” from each of the four new elements), (iii) further elements may have to be refined until the 2-to-1 property is fulfilled, i.e. all element vertices are shared by other element vertices and/or in the center of element edges.

It is noted that the four sub-elements resulting from an element refinement replace their parent element. One may associate the term “refinement level” with each element indicating the number of refinements that have been realized until it was “born”. Clearly, different refinement levels are present in each collection of elements \( P_l \). It is a direct consequence of (iii) that all neighboring elements of each element have a maximum difference in the refinement level of 1.

For the final collection of elements, \( P_{n_{ref}} \), let us now define nodes which will be associated with shape functions later on. So far we have only used the expression vertex which is related rather to the geometry than to approximations. Especially for higher-order shape functions on 1-irregular meshes, nodes and vertices are clearly not the same. Only for bi-linear elements as considered here, nodes \( i \) with coordinates \( x_i \) are placed right at the vertices. The nodal set \( I_h \) involves all the nodes in the domain. Furthermore, the hanging nodes are given in the subset \( I_h \subseteq I \). All other nodes are called regular nodes and are in the set \( I_r = I \setminus I_h \). It is useful to associate a pair (2-tuple) with each hanging node which defines the other two regular nodes.
that share the edge with the hanging node:

\[ Q_k = \{(i, j) : \text{nodes } i, j \in I_r \text{ share the edge with hanging node } k\}, k \in I_h. \] (1)

2.2. Refinement criteria

As noted above, each refinement step \( l + 1 \) starts with the marking of elements for (further) refinement, i.e., the construction of an element set \( W^l_{\text{ref}}, 0 \leq l \leq n \). This can either be achieved based on mathematical considerations in the context of error estimation or based on \textit{a priori} knowledge of the solution. Examples for the latter case are refinements near corners where singularities are expected or refinements in order to increase the resolution of interfaces.

2.2.1. Refinement at interfaces

For applications, where steep gradients are expected near an interface or where the resolution of the interface is crucial for the overall solution, the mesh refinement is realized in the vicinity of the interface. In this work, an interface \( \Gamma_d \) is defined implicitly by means of the level-set method [44, 43, 46], which has become a standard in the context of the XFEM. The signed distance function is used as a particular level-set function

\[ \phi(x) = \pm \min_{x^* \in \Gamma_d} \| x - x^* \| \] (2)

for all \( x \in \Omega \), where \( \| \cdot \| \) denotes the Euclidean norm. Consequently, the zero-isocontour of the level-set function is the interface, i.e., \( \phi(x) = 0 \) for all \( x \in \Gamma_d \). The sign is different on the two sides of the interface and can be determined from \( \text{sign}(n \cdot (x - x^*)) \) with \( x^* \) being the closest point on the interface to \( x \). The level-set function \( \phi(x) \) is typically given at nodes and interpolated by means of standard FE shape functions (we still write \( \phi \) rather than \( \phi^h \) for brevity).

Assume that a refinement of all elements within the distance \( d \in \mathbb{R}^+ \) from the interface is desired, then

\[ W^l_{\text{ref}} = \left\{ k \in P_l : \max_{x \in \omega^l_k} |\phi(x)| < d \right\}. \] (3)

It may also be useful to refine only the set of elements cut by the interface,

\[ W^l_{\text{ref}} = \left\{ k \in P_l : \min_{x \in \omega^l_k} \phi(x) \cdot \max_{x \in \omega^l_k} \phi(x) < 0 \right\}. \] (4)

2.2.2. Refinement at points (singularities)

Let us assume a point-wise steep gradient such as a singularity at a point \( x' \). The mesh refinement is then useful for elements in the vicinity of \( x' \),

\[ W^l_{\text{ref}} = \left\{ k \in P_l : \min_{x \in \omega^l_k} \| x - x' \| < d \right\}. \] (5)

Furthermore, one may only want to refine the element that contains \( x' \),

\[ W^l_{\text{ref}} = \left\{ k \in P_l : x' \in \omega^l_k \right\}. \] (6)
2.2.3. Refinement by error estimators For simplicity, we do not consider local mesh refinements based on (a posteriori) error estimators in this work. Error estimators are not only a wide topic in the context of the classical finite element method, see e.g. [3, 54], but adjustments may need to be considered in the context of hanging nodes, see Ainsworth et al. [3, 2], and also the XFEM, see Bordas et al. [11, 10, 25] and Rodenas et al. [45]. It is thus expected that error estimators that are suited for both, meshes with hanging nodes and extended approximations as used in the XFEM, may deserve special care. Examples where error estimators in hp-adaptive schemes are applied in a straightforward manner to XFEM and hanging nodes are found in [16, 14, 15].

3. General Formulation of the XFEM

A standard extended finite element approximation of a function \( u(x) \) is of the form

\[
\begin{align*}
  u^h(x) &= \sum_{i \in I} N_i(x) u_i + \sum_{i \in I^*} N_i(x) \psi(x) a_i, \\
  \text{strd. FE approx.} &\quad \text{enrichment}
\end{align*}
\]

(7)

where for our purposes only one enrichment term is considered. The approximation consists of a standard finite element (FE) part and the enrichment. \( N_i(x) \) are the shape functions which will be specified in the two following sections for meshes with hanging nodes. In the XFEM, it is crucial that these functions build a partition of unity in the enrichment part. The nodal coefficients of the standard FE and enriched part are denoted as \( u_i \) and \( a_i \), respectively. \( I^* \subset I \) is the set of enriched nodes and \( \psi(x) \) is the problem-dependent enrichment function. It is often useful to “shift” the enrichment part in (7) by using \( [\psi(x) - \psi(x_i)] \) instead of \( \psi(x) \) in order to maintain the Kronecker-\( \delta \) property of the standard FE part of the approximation [9, 29].

Herein, only enrichment functions for jumps and kinks are considered. For jumps, also referred to as strong discontinuities, the Heaviside-enrichment is a frequent choice, i.e.

\[
\psi(x) = H(\phi(x)) = \begin{cases} 
0 : & \phi(x) \leq 0, \\
1 : & \phi(x) > 0.
\end{cases}
\]

(8)

For kinks, also referred to as weak discontinuities, enrichment functions based on the absolute value of the level-set function are often chosen, for example \( \psi(x) = |\phi(x)| \) (“abs-enrichment”). It is well-known that in this case some enrichment functions cause problems in blending elements that may decrease the overall convergence rate, see e.g. Chessa et al. [18] and Fries et al. [17, 27, 29]. Herein, we employ a modification of the pure abs-enrichment as proposed by Moës et al. [39],

\[
\psi(x) = \sum_{i \in I^*} N_i(x) \cdot |\phi(x_i)| - \sum_{i \in I^*} N_i(x) \cdot |\phi(x_i)|.
\]

(9)

The enrichment functions (8) and (9) for jumps and kinks, respectively, are independent of the considered model equations. This is in contrast to most enrichment functions for steep gradients which are often tailored for specific models. An example are the enrichment functions for cohesive or cohesionless cracks, see e.g. Meschke and Dumstorf [37], Moës and Belytschko.
An enrichment scheme for steep gradients which is independent of the model under consideration is proposed in Abbas et al. [1]. The idea is to employ a set of enrichment functions that covers the complete range of gradients starting from those which can no longer be captured by the standard FE part of the approximation up to a true discontinuity. An adaptive procedure for adjusting enrichment functions is proposed in Waisman and Belytschko [58].

The choice of the enriched nodes \( I^\star \) and the definition of the shape function \( N_i(x) \) is discussed in the following sections.

4. Hanging nodes with DOFs

We distinguish two approaches depending on whether DOFs are associated with the hanging nodes or not. The extension of the XFEM for the case of hanging nodes with DOFs is particularly simple. The key aspect is the construction of a conforming finite element space on the 1-irregular mesh which builds a partition of unity in the domain. In fact, the PU-property is only needed where the enrichment in the XFEM is to be realized, however, as shall be seen below, the function space used herein builds a PU in the whole domain. Shape functions for two-dimensional 1-irregular meshes are proposed by Gupta in [32], and for three dimensions by Morton et al. [41]. For general \( k \)-irregular meshes, Legendre-type shape functions are proposed by Baitsch and Hartmann in [5] and Karniadakis and Sherwin [33]. An approach by Cho et al. [20, 19] constructs shape functions for non-matching interface discretization which may also be understood as hanging nodes. For a comparison of higher-order Lagrange-type and higher-order Integrated Legendre or Gauss-Lobatto shape functions in the context of the XFEM, see Byfut et al. [15].

4.1. Conforming shape functions

The shape functions used herein have been proposed by Gupta in [32]. The definition is based on the reference element \( \Omega^\star = (-1,1) \times (-1,1) \) with 4 corner nodes and 4 (potential) hanging nodes in the element centers, see Figure 3(a). The reference shape functions associated with the hanging nodes are defined as

\[
N_5^\star = \frac{1}{2} (1 - |\xi|) \cdot (1 - \eta) \\
N_6^\star = \frac{1}{2} (1 + \xi) \cdot (1 - |\eta|) \\
N_7^\star = \frac{1}{2} (1 - |\xi|) \cdot (1 + \eta) \\
N_8^\star = \frac{1}{2} (1 - \xi) \cdot (1 - |\eta|)
\]

for all \( \xi, \eta \in \Omega^\star \). If any of the hanging nodes is not present in a particular element of the collection \( \mathcal{P} \), the corresponding shape function is set to zero. We say an element has no hanging nodes if no nodes exist in the edge centers; we do not relate this to whether the corner node of an element is a hanging node in its neighbor element, see Figure 3(b).

The original bi-linear shape functions at the corner nodes 1–4 are modified in the following
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Figure 3. (a) Node numbering in the reference element. (b) Although the corner node of the small element is a hanging node, it is called a “standard element without hanging node”. Only elements with hanging nodes on their edge centers are called “elements with hanging nodes”.

\begin{align*}
N_1^* &= \frac{1}{4} (1 - \xi) \cdot (1 - \eta) - \frac{1}{2} (N_6^* + N_8^*) \\
N_2^* &= \frac{1}{4} (1 + \xi) \cdot (1 - \eta) - \frac{1}{2} (N_6^* + N_7^*) \\
N_3^* &= \frac{1}{4} (1 + \xi) \cdot (1 + \eta) - \frac{1}{2} (N_6^* + N_7^*) \\
N_4^* &= \frac{1}{4} (1 - \xi) \cdot (1 + \eta) - \frac{1}{2} (N_7^* + N_8^*)
\end{align*}

The corresponding element shape functions in the physical domain \( \Omega \) are obtained by a bi-linear mapping from the reference domain. The shape functions \( N_i(x) \) for each node at \( x_i \), result from the usual assembly of the element shape functions. A visualization of some shape functions \( N_i(x) \) is shown in Figure 4. Because some shape functions have a kink on the \( \xi \) or \( \eta \)-axis, a special quadrature is required in elements with hanging nodes: Gauss points are placed in 4 sub-elements \((-1,0) \times (-1,0), (0,1) \times (-1,0), (0,1) \times (0,1), \) and \((-1,0) \times (0,1)\) of the reference element, see Figure 6(a).

It is trivial to show that the resulting shape functions are conforming and build a PU over \( \Omega \). Furthermore, the extension of these shape function definitions to higher-order and/or three dimensions is straightforward as already noted in [32].

4.2. Coupling with XFEM

4.2.1. Choice of the enriched nodes  Herein, the XFEM is used in order to capture jumps or kinks at \( \Gamma_d \). The position of \( \Gamma_d \) is specified by the level-set function \( \phi(x) \). Let \( W_{\text{cut}} \) be the collection of elements that are cut by \( \Gamma_d \),

\[
W_{\text{cut}} = \left\{ k \in \mathcal{P}_{\text{cut}} : \min_{i \in I_k^\text{el}} \phi(x_i) \cdot \max_{i \in I_k^\text{el}} \phi(x_i) < 0 \right\},
\]

where \( I_k^\text{el} \) are the element nodes of element \( k \). For 1-irregular meshes as defined in Section 2.1, \( I_k^\text{el} \) is an \( n \)-tuple with \( 4 \leq n \leq 8 \).
Figure 4. Shape functions of the “hanging nodes with DOFs”-approach. Note that the shape functions in (a) and (b) are zero over parts of the surrounding elements.

For cracks, $\Gamma_d$ represents the crack path and ends at the crack-tip at $x'$. This can be described in the level-set context by using two level-set functions as proposed by Stolarska et al. [50, 49]. Then, $W_{\text{cut}}$ involves only those elements that are entirely cut by the crack. It is noted that the pure Heaviside-enrichment (8) does not correctly capture the crack-tip at $x'$ but virtually extends the crack to the next element edge. This is well-known in the context of the XFEM and the situation may be solved by using approaches such as discussed by Zi and Belytschko [59]. On the other hand, the refinement at the crack-tip that is used in this paper also improves the situation significantly so that advanced approaches are not employed.

Based on $W_{\text{cut}}$, the set of enriched nodes $I^*$ is defined as

$$ I^* = \bigcup_{k \in W_{\text{cut}}} I_{k, \text{el}}. \quad (19) $$

In words, those nodes are enriched that belong to elements entirely cut by $\Gamma_d$. See Figure 5 for examples of $I^*$.
4.2.2. Quadrature  As noted above, a special integration in elements with hanging nodes is needed even without any enrichment. In the context of the XFEM one has to furthermore consider that the shape functions inherit the continuity-properties of the corresponding enrichment function $\psi(x)$. That is, for the Heaviside-enrichment (8), the shape functions are strongly discontinuous at $\Gamma_d$, and weakly discontinuous for the (modified) abs-enrichment (9). This must be considered in the quadrature such as shown in Figure 6(b): The reference element is decomposed into four sub-elements as mentioned above. Linear interpolation of the level-set function is used in two triangles that decompose the reference element. It is then ensured that the interface is piecewise straight and the quadrature appropriately accounts for the discontinuities.

Figure 6. Integration points in a reference element with hanging nodes.

4.2.3. Blocking of enriched DOFs  Reference shape functions in the element corners, i.e. $N_1^* - N_4^*$, may be zero over parts of the reference elements for some hanging node situations. Depending on the enrichment function $\psi(x)$, this may lead to a shape function $N_i(x) \cdot \psi(x)$ which is zero everywhere! It is thus important to check for zero-shape functions and eliminate the corresponding enriched DOF from the system of equations (i.e. from $I^*$). This issue is not related to the well-known problems with ill-conditioning in the XFEM as, e.g., discussed by Bordas et al. [12], Liu et al. [36], and Daux et al. [22].

4.3. Properties.

The properties of this approach are summarized as follows:

- Hanging nodes have DOFs and associated shape functions.
- The support of each regular or hanging node is the union of elements that share this node.
- Even without any enrichment, elements with hanging nodes have special shape functions and the dimension of the element matrices is variable. Special integration is required which places Gauss points in four sub-elements of the reference element.
- Elements without hanging nodes are treated as standard FEM or XFEM-elements.
- The choice of the enriched nodes is very intuitive and is consistent with the situation for XFEM on conforming meshes. Sometimes, enriched nodes have to be blocked if they do not contribute to the system matrix.

5. Hanging nodes without DOFs (constrained approximation)

The (standard) treatment of hanging nodes may be summarized as follows: In a first step, hanging nodes are treated as regular nodes and the usual algorithms for the assembly of the system matrix are pursued. In a second step, constraints for DOFs associated with hanging nodes are introduced. This approach is frequently referred to as constrained approximation, see e.g. Ainsworth et al. [4, 3, 2].

Depending on how the constraints are enforced one may distinguish two approaches; both approaches yield identical results. In the following, the approaches are first described in a classical FEM context (i.e. without enrichment) and then extended to enriched approximations.

5.1. Lagrange multipliers and connectivity matrices

5.1.1. Constraints at hanging nodes

Each hanging node $k \in I_h$ is constrained to be the average of its neighboring corner nodes, i.e.,

$$ u_k = \frac{1}{2} u_i + \frac{1}{2} u_j \quad \text{for } i, j \in Q_k. \tag{20} $$

For the definition of $I_h$ and $Q_k$, see Section 2.1. One may write equation (20) in vector form as

$$ \gamma_k \cdot u = 0, \tag{21} $$

with $\gamma_k$ being a zero-vector with 1 at position $k$ and $-1/2$ at positions $i, j \in Q_k$, e.g.

$$ \gamma_k = [0 \ldots \underbrace{0}_{i} - \frac{1}{2} \underbrace{0}_{k} \ldots 0 \underbrace{0}_{j} \ldots 0]^T. \tag{22} $$

Let $C$ be the matrix of all vectors $\gamma_k$, $k \in I_h$,

$$ C = [\gamma_1, \ldots, \gamma_k], \tag{23} $$

where the dimension of $C$ is the number of all nodes times the number of hanging nodes. Furthermore, we introduce a global connectivity matrix $\Pi$ which is composed by vectors $\pi_k$. For $k$ being a regular node, $k \in I_r$, $\pi_k$ is a zero vector with a 1 at position $k$. For $k$ being a hanging node, $k \in I_h$, $\pi_k$ is a zero vector with $1/2$ at positions $i, j \in Q_k$, e.g.,

$$ \pi_k = [0 \ldots 0 \underbrace{\frac{1}{2}}_{i} 0 \ldots 0 \underbrace{\frac{1}{2}}_{k} 0 \ldots 0]^T. \tag{24} $$

The square matrix $\Pi^*$ consists of all vectors $\pi_k$, $k \in I$,

$$ \Pi^* = [\pi_1, \ldots, \pi_k]. $$
The rows belonging to the hanging nodes are zero in this matrix. The connectivity matrix $\Pi$ is then obtained by only considering the rows of the regular nodes in $\Pi^\star$ and omitting the zero-rows for the hanging nodes. The dimension of $\Pi$ is the number of regular nodes times the number of all nodes.

**Remark.** In applications with constrained approximation, one would typically not assemble the matrices $C$ and $\Pi$ explicitly but only work with local or element-wise defined connectivity matrices related to the existing nodes in a given element, cf. [33].

### 5.1.2. Non-conforming shape functions

In a first step, we assume that the shape functions are only the standard bi-linear shape functions on the 1-irregular mesh. Only the corner nodes of each element contribute to the element shape functions. That is, a standard bi-linear reference element is employed with $N_1^\star, \ldots, N_4^\star$ as defined in (14) – (17) with $N_5^\star = N_6^\star = N_7^\star = N_8^\star = 0$. The resulting shape functions $N_i^b$ in the domain are clearly non-conforming, see Figure 7. Thus, the conformity must be explicitly enforced through the constraints (20).

![Figure 7. Standard bi-linear shape functions are non-conforming along edges with hanging nodes.](image)

We assume that the following system of equations results from a standard finite element assembly based on the non-conforming shape functions,

$$ Au = b. $$

(25)

We are now concerned with the enforcement of the constraints.

### 5.1.3. Lagrange multipliers

In this approach, a Lagrange multiplier $\lambda_k$ is introduced for each hanging node $k \in I_h$. The constraint (20) is then imposed by

$$ \lambda_k \cdot (u_k - \frac{1}{2} u_i - \frac{1}{2} u_j) = 0. $$

(26)

The original system of equations (25) resulting from the use of the shape functions as described in Section 5.1.2 is then altered as follows,

$$ \begin{bmatrix} A & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}. $$

(27)
It is noted that the equations for the Lagrange multiplier may have to be scaled to maintain reasonable condition numbers for the altered system of equations.

5.1.4. System reduction

The augmentation (27) of the original system of equations (25) is not necessary for constraints as given by (20). In fact, the system may even be reduced using the global connectivity matrix $\Pi$ defined in Section 5.1.1, such that only

$$\Pi \cdot A \cdot \Pi^T \cdot u_r = \Pi \cdot b,$$

(28)

has to be solved. The final solution providing appropriate nodal coefficients for all nodes is then given by

$$u = \Pi^T u_r.$$  

(29)

It is noted that the nodal coefficients $u$ resulting from the augmented and from the reduced system of equations are identical.

5.1.5. A note on the extension to XFEM

Consider the situation depicted in Figure 8 where nodes A, B, C, D have to be enriched due to the presence of the interface in element 1. Consequently, at the hanging node D, the standard unknown $u_D$ and the XFEM unknown $a_D$ are present. The imposition of constraint (20) for $u_D$ is straightforward, i.e.,

$$u_D = 0.5 \cdot u_C + 0.5 \cdot u_E$$  

(30)

and may thus be easily enforced via the Lagrange multiplier or the system reduction technique. In order to be able to constrain $a_D$ similarly, it is necessary to enrich also node E, such that we may impose the constraint

$$a_D = 0.5 \cdot a_C + 0.5 \cdot a_E.$$  

(31)

It shall be seen in the following Section that this “additional” enrichment is well justified.

Figure 8. In constrained approximation, one has to get rid of the standard and enriched DOF at the hanging node D.

5.2. Conforming shape functions and XFEM

The viewpoint taken in Section 5.1 is very much based on the idea to employ the usual bi-linear shape functions for each node and to constrain contributions to the system matrix from all DOFs associated to hanging nodes afterwards. However, if we apply the constraints (20) to the non-conforming shape functions $N^h_i = N^h$ in the first place, we arrive at a different though
equivalent viewpoint. The resulting shape functions, in terms of the global connectivity matrix \( \Pi \),

\[
N = \Pi \cdot N^b.
\]  

(32)

are conforming on the 1-irregular mesh, are only present at regular nodes, and build a partition of unity in the domain. An example for these shape functions is depicted in Figure 9. A comparison with Figure 4 reveals that the functional form and support differs from the approach described in Section 4.

Figure 9. Shape functions of the “hanging nodes without DOFs”-approach. No shape function exists for the hanging node in (c).

Due to the partition of unity property, it is natural to extend the XFEM to this approach. Hence, following the usual strategy, all nodes whose associated shape functions are entirely cut by the crack/interface have to be enriched, cf. [40]. As there is no shape function associated to a hanging node in this viewpoint, hanging nodes are not enriched. On the other hand, the (regular) nodes neighboring a hanging node are enriched if the support of their associated shape functions are entirely cut. This creates an additional step in the simulation codes which is not present in the approach with DOFs at hanging nodes. As a result, the set of enriched nodes from this approach will differ from the one given in Section 4.2.1.

Going back to the viewpoint of Section 5.1, it become clear that all hanging nodes and their neighboring (regular) nodes have to be enriched in the first step. In the second step, regular and enriched DOFs associated to hanging nodes only have to be constrained as explained in Sections 5.1.3 and 5.1.4. Given this insight, the extension of the matrices \( C \) and \( \Pi \) for the combination of constrained approximations and the XFEM is straightforward.

5.2.1. Shifted enrichments and constrained approximation

It is important to note that the combination of shifted enrichments and constrained approximation requires special care. Shifted enrichments are standard in many XFEM simulations as they facilitate the application of boundary conditions. One may show that the standard procedure to first assemble the system of equations for the non-conforming shape functions and then apply the constraints (30) and (31) is not always suitable for the case of shifted enrichments. Consider the situation depicted
in Figure 10. For example, nodes $x_1$, $x_2$, and $x_3$ may correspond to nodes $C$, $E$, and $D$ in Figure 8.

![Figure 10. One-dimensional shape functions for an edge with a hanging node.](image)

The corresponding shape functions are

\[
N_1(x) = \frac{1}{2}(1 - x), \quad N_2(x) = \frac{1}{2}(1 + x),
\]

\[
M_1(x) = \begin{cases} 
-x & x \in [-1, 0], \\
0 & \text{otherwise},
\end{cases} \quad M_2(x) = \begin{cases} 
x & x \in [0, 1], \\
0 & \text{otherwise},
\end{cases}
\]

\[
M_3(x) = \begin{cases} 
1 + x & x \in [-1, 0], \\
1 - x & x \in [0, 1].
\end{cases}
\]

for $x \in [-1, 1]$. There obviously holds

\[
N_1(x) = M_1(x) + \frac{1}{2}M_3(x), \quad (33)
\]

\[
N_2(x) = M_2(x) + \frac{1}{2}M_3(x) \quad (34)
\]

for all $x \in [-1, 1]$ which refers to the constraints given in (30). Let $\psi_i(x)$ be some shifted enrichment function, i.e.,

\[
\psi_i(x) := \hat{\psi}(x) - \hat{\psi}(x_i),
\]

for some fixed $x_i$, $i \in \{1, 2, 3\}$ and some enrichment function $\hat{\psi}(x)$, e.g., the Heaviside-enrichment given in (8). Using (33) and (34), the nodal enrichment functions for $x_1$ and $x_2$ may be expressed as

\[
\psi_1(x)N_1(x) = \psi_1(x)M_1(x) + \frac{1}{2}\psi_1(x)M_3(x),
\]

\[
\psi_2(x)N_2(x) = \psi_2(x)M_2(x) + \frac{1}{2}\psi_2(x)M_3(x)
\]

for all $x \in [-1, 1]$. These enrichment functions are obviously not obtained if nodal enrichment functions are first constructed based on $M_i(x)$, i.e.,

\[
\psi_1(x)M_1(x), \quad \psi_2(x)M_2(x), \quad \psi_3(x)M_3(x), \quad (35)
\]

and then applying (31) resulting in

\[
\psi_1(x)N_1(x) = \psi_1(x)M_1(x) + \frac{1}{2}\psi_3(x)M_3(x), \quad (36)
\]

\[
\psi_2(x)N_2(x) = \psi_2(x)M_2(x) + \frac{1}{2}\psi_3(x)M_3(x), \quad (37)
\]
unless
\[ \psi_1(x) = \psi_2(x) = \psi_3(x). \]  
(38)

For the step-enrichment, condition (38) is met as long as the element edge with the hanging node is not cut. Furthermore, the problem does not appear when un-shifted enrichment functions are applied as the enrichment function is the same. It is now clear that for the cases considered herein, special care is required for constrained approximation and shifted enrichments if interfaces cut through elements with hanging nodes. In these cases, it is important to construct the set of conforming shape functions first rather than using the non-conforming shape functions and apply (30) and (31) afterwards.

5.3. Properties

The properties of this approach—that employs no DOFs at hanging nodes—are summarized as follows:

- Hanging nodes have no DOFs and no associated (conforming) shape functions.
- The support of a regular node \( k \) is the union of elements that share this node or the potential hanging nodes on the edges from node \( k \).
- Without any enrichment, elements with hanging nodes are treated as standard elements, i.e. only the bi-linear shape functions at the 4 corner nodes are considered and the element matrices have a constant dimension.
- Elements without hanging nodes in the edge centers but in the element corners have modified shape functions. The hanging node is replaced by the corresponding regular nodes on the same edge.
- The choice of the enriched nodes needs to consider that hanging nodes have no DOF, so that instead of the hanging node, the regular nodes that share this edge are enriched.

6. Numerical results

Numerical results from solid and fluid mechanics are presented. The performance of the XFEM on meshes with hanging nodes is shown where the refinement is realized in order to (i) increase the resolution of the interface representation and/or (ii) capture general steep gradients near the interface. In solid mechanics, the test case discussed in Section 6.1.2, involves a material interface. No steep gradients are present in the displacement fields but the refinement increases the resolution of the curved interface. Crack problems are discussed in Sections 6.1.3 and 6.1.4. Only straight cracks are considered so that the resolution of the crack path is not an issue. The refinement is realized near the crack-tips where the stresses and strains are singular. Section 6.2 shows applications where both aspects for the mesh refinement—the resolution and steep gradients—are of importance. It is noted that the fluid test cases are instationary and time-integration schemes are required; see Fries and Zilian [30] for a discussion of time integration in the context of the XFEM.
6.1. Applications in solid mechanics

6.1.1. Governing equations of linear elasticity

The strong form for an elastic solid in two dimensions, undergoing small displacements and strains under static conditions, is

\[ \nabla \cdot \sigma = f, \quad \text{on } \Omega \subseteq \mathbb{R}^2, \quad (39) \]

where \( f \) describe volume forces, and \( \sigma \) is the stress tensor,

\[ \sigma = C : \varepsilon = \lambda (\text{tr} \varepsilon) \mathbf{I} + 2\mu \varepsilon, \quad (40) \]

with \( \lambda \) and \( \mu \) being the Lamé constants, cf. \[8, 60\]. The linearized strain tensor \( \varepsilon \) is

\[ \varepsilon = \frac{1}{2} \left( \nabla d + (\nabla d)^T \right) \quad (41) \]

and involves gradients of the displacements \( d \). The boundary of the domain \( \Omega \) is separated into two disjoint parts \( \partial \Omega = \Gamma_d \cup \Gamma_t \). Dirichlet boundary conditions are prescribed displacements \( \hat{d} \) on \( \Gamma_d \), i.e.,

\[ d = \hat{d}, \quad \text{on } \Gamma_d. \quad (42) \]

Neumann boundary conditions are applied on \( \Gamma_t \) as

\[ \sigma n = \hat{t}, \quad \text{on } \Gamma_t, \quad (43) \]

where \( n \) is the outer normal vector of the boundary. Furthermore, either a material interface \( \Gamma_d \) or a stress-free crack path \( \Gamma_d \) is present in \( \Omega \) (only edge crack problems are considered in this paper). Enriched approximations of the form \( (7) \) are used for each of the displacement components in \( d \). In the case of a material interface, the abs-enrichment \( (9) \) is chosen. For crack problems, the Heaviside-enrichment \( (8) \) is employed at the crack surface.

For brevity, we do not show the weak form of this problem which can be found in numerous references such as \[8, 60\]. It is only noted that the test and trial spaces involve classical finite element shape functions and enrichment functions that are discontinuous at the interface/crack path.

6.1.2. Bi-material problem

The aim of the first test case is to improve the results by increasing the resolution of the interface. An un-cracked domain with a circular material interface \( \Gamma_d \) is considered. No steep gradients are present in the displacement fields. The situation is depicted in Figure 11(a). A weak discontinuity is present at \( \Gamma_d \), i.e., the displacement field is continuous with discontinuous stresses and strains. Inside the circular plate of radius \( b \), whose material is defined by \( E_1 = 10 \) and \( \nu_1 = 0.3 \), a circular inclusion with radius \( a \) of a different material with \( E_2 = 1 \) and \( \nu_2 = 0.25 \) is considered. The loading of the structure results from a linear displacement of the outer boundary: \( u_1(b, \theta) = r \) and \( u_2(b, \theta) = 0 \). The exact solution may be found, e.g., in \[27, 52\] and is not repeated here.

For the numerical model, the domain is a square of size \( L \times L \) with \( L = 2 \). The outer radius is chosen to be \( b = 2 \) and the inner radius \( a = 0.4 + \epsilon \). The parameter \( \epsilon \) is set to \( 10^{-3} \), and avoids that the levelset function is exactly zero at a node of a given mesh. (In this case, the discontinuity would directly cut through that node). The exact stresses are prescribed along the boundaries of the square domain, and displacements are prescribed as \( u_1(0, \pm 1) = 0 \) and \( u_2(\pm 1, 0) = 0 \). Plane strain conditions are assumed. For the XFEM simulation, the enrichment function \( (9) \) is used.
We use different meshes with \( n_{\text{el}} = \{5, 10, 20, 40, 80\} \) elements in each dimension. The local refinement at the interface is pursued with respect to criterion (4) with a different maximum number of refinement levels of \( n_{\text{ref}} = \{1, 2, 3, 4, 5\} \). Results are shown Figure 12 for the two different approaches for the hanging nodes. Recall that the refinement for this test case only increases the resolution of the curved interface and no steep gradients near the interface exist. It is seen that an optimal convergence rate of 2 in the \( L_2 \)–norm is asymptotically achieved in all cases. Furthermore, from a certain refinement level, the error is no longer dominated by the resolution of the interface but on the approximation error in the overall domain. Then, further refinement near the interface does not improve the results. Clearly, the coarser the original mesh, the more refinement levels improve the solution considerably. Finally, comparing Figures 12(a) and (b), it is seen that the two approaches yield very similar results. The slightly better results (by a constant) for the case where DOFs at the hanging nodes are considered is easily explained by the reason that more DOFs improve the solution.

6.1.3. Pure mode I and II edge crack A square domain of size \( L \times L \) is considered with an edge-crack of length \( a \), see Figure 13(a) for a sketch. Along the boundary of the square domain, displacements are prescribed such that the well-known analytic solution of a near-tip crack field is the exact solution in the entire domain [26, 27]. That is, the overall displacement field is defined by a linear combination of crack mode I and II, see Figure 13(b) and (c), scaled by the stress intensity factors \( k_1 = 2 \) and \( k_2 = 3 \). The material is defined by \( E = 10000 \) and \( \nu = 0.3 \).

For the numerical computations, we choose \( L = 2, a = 1 \). Plane stress conditions are assumed. Structured initial meshes are used with \( n_{\text{el}}^0 = \{19, 29, 39, 59, 99, 129, 159\} \) elements per dimension which are then refined. A mild perturbation is added to the nodes of the initial meshes (in the range of \( 10^{-4} \)) in order to avoid that the crack coincides with element edges. Results are obtained on meshes that are locally refined according to criterion (6), see Figure
Figure 12. Convergence results for the bi-material problem for the approach (a) with and (b) without DOFs at the hanging nodes.

Figure 13. a) Problem statement of the edge-crack problem, b) displacements and von-Mises stresses according to the mode I and II crack.

Figure 14. Meshes used for the crack test case: (a) uniform mesh, (b) locally refined with hanging nodes, (c) perturbed refined mesh with hanging nodes.
14(b), and the two different approaches for the hanging nodes. For these two cases, only the step-enrichment is employed. It is important to note that the crack cuts through elements with hanging nodes. This is in contrast to situations where the refinement is realized with respect to a closed interface. There, the elements cut by the interface never have hanging nodes.

We compare the two approaches with hanging nodes to results that are obtained on uniform meshes, see Figure 14(a), where the so-called “branch enrichment” is used in order to capture the steep gradients at the tip, see e.g. [40]. Along the crack-path, the step enrichment is employed. Two settings are considered for the branch enrichment: One where only the crack-tip element is enriched and another where the enrichment is realized in a fixed radius around the crack-tip [35, 27]. It is noted that the number of integration points for the pure step-enrichment is not increased at the crack-tip whereas a large number of points is used where the branch enrichment is employed. Thus, the approaches with the branch enrichment are considerably more time-consuming for the same number of DOFs in the overall system of equations than the pure step-enrichment.

![Graphs showing convergence results for different cases](image)

Figure 15. Convergence results for the crack test case in a square domain: (a) convergence in the $L_2$-norm; (b) and (c) convergence to the exact stress intensity factors of $k_1 = 2$ and $k_2 = 3$, respectively; (d) convergence in the $L_2$-norm for different refinement levels.

Results are shown in Figure 15(a) for the $L_2$-norm of the displacements. For the refined
meshes with hanging nodes, 5 refinement levels are realized. It is seen that approach 1 with DOFs at the hanging nodes is somewhat more accurate than approach 2 (constrained approximation). As expected, the convergence rate is 1 in the $L_2$-norm for both approaches as no special enrichments consider for the singularity of the stresses and strains at the crack-tip [51]. The reason why the difference between the two approaches is more significant than for the previous test case is that the additional degrees of freedom for approach 1 are placed in the vicinity of the crack-tip, i.e., in the region which dominates the approximation error. Every additional DOF in this region has the potential to improve the solution noticeably.

Compared to the case with the branch enrichment in the crack-tip element, it is interesting to note that the results with the pure step-enrichment on a refined mesh are more accurate. That is, the additional effort for the implementation and computation of the branch enrichment does not pay off as long as only the crack-tip element is enriched. On the other hand, it is seen that enriching with the branch enrichment within a constant radius around the crack-tip leads to optimal convergence rates of 2 in the $L_2$-norm and results are much better than with all other approaches. However, it is noted that branch enrichments within a constant radius show problems for curved and/or propagating cracks. Therefore, it is still standard to enrich only the crack-tip element. Then, as could be seen, a pure step-enrichment on locally refined meshes clearly has the potential to lead to more accurate results. Figures 15(b) and (c) confirm these findings for the convergence to the exact stress intensity factors of $k_1 = 2$ and $k_2 = 3$. Stress intensity factors have been computed by means of the interaction integral as, e.g., described in [40].

Let us now compare the results on meshes with hanging nodes for different refinement levels. Results in the $L_2$-norm are shown in Figure 15(d) for the approach with DOFs at the hanging nodes. Compared to the situation with the branch enrichment only in the crack-tip element, it is found that already a refinement-level of 3 yields more accurate results. Results on randomly perturbed meshes, see Fig. 14(c), are shown as well: it is seen that the convergence is not as smooth as for unperturbed meshes yet, in average, the same convergence rate is achieved.

Finally, also the condition number and the total number of DOFs in the different approaches are discussed. We have confirmed that the condition number of both approaches with hanging nodes and for the case of the branch enrichment in the crack-tip element is in the same range of $\approx 10^7$ for initial meshes with up to 99 elements in each dimension. On the other hand, it is well-known that the condition number for the branch enrichment within a constant radius grows dramatically with mesh refinement [35], and is already $\approx 10^9$ for only 39 elements in each dimension.

The total number of DOFs associated with the different approaches is dominated by $\left( n_{\text{el}}^1 \right)^2$. The enrichments add DOFs scaled by $n_{\text{el}}^3$, except for the case of the branch enrichment within a constant radius where it is scaled by $\left( n_{\text{el}}^3 \right)^2$. It is interesting that the local refinement with respect to the crack-tip adds a constant number of additional DOFs independent of $n_{\text{el}}^3$ (slightly more for the approach with DOFs at the hanging nodes). Therefore, when plotting the results over the DOFs rather than over the element size $h$, the same conclusions from above apply.

6.1.4. Shear and tensile edge crack. The next example considers a rectangular domain of dimensions $L \times W$ with an edge crack of length $a$, see Figure 16(a). Following [7], we choose $L = 16$, $W = 7$, and $a = 3.5$. The structure is clamped at the bottom and is loaded by $q_x$ and $q_y$ on the top. Plane strain conditions are assumed. The Poisson’s ratio is 0.3 and Young’s modulus is 100.
Two settings are considered: In the first case, a tensile edge crack is present due to \( q_x = 0 \) and \( q_y = 1 \). The exact stress intensity factor is defined as \( k_1 = C(a/W) \cdot q_y \cdot \sqrt{\alpha \pi} \) with

\[
C(a/W) = 1.12 - 0.231(a/W) + 10.55 \cdot (a/W)^2 - 21.72 \cdot (a/W)^3 + 30.39 \cdot (a/W)^4,
\]

(44)

cf. [26]. For the parameters chosen here, \( k_1 = 9.372139 \). In the other case, a shear edge crack results from \( q_x = 1 \) and \( q_y = 0 \). The exact stress intensity factor is then \( k_1 = 34.0 \) as given, e.g., in [7].

Initial meshes have \( \{11, 21, 31, 51\} \) elements in \( x \)-direction and \( \{21, 41, 61, 101\} \) elements in \( y \)-direction. The convergence to the exact values of the stress intensity factor \( k_1 \) for the tensile and shear edge crack is given in Figure 16(b) and (c), respectively. It is seen that previous findings are confirmed: Obviously, the higher the refinement level, the better are the results. In contrast to the bi-material test case discussed in Section 6.1.2, even the refinement from level 4 to level 5 improves the results noticeably. This indicates that the refinement is realized where it is needed, namely at the crack-tip. For this test case, only results from the approach with DOFs at the hanging nodes are shown here as constrained approximation yields very similar results. Furthermore, it is seen that already a level-3-refinement again leads to more accurate results than obtained with a branch-enriched crack-tip element on a uniform mesh.
6.1.5. Angled crack  This test case is taken from [36]. Assume a square domain of dimensions $W \times W$ with an angled crack of length $2 \cdot a$, see Figure 17(a). We choose $W = 10$ and $a = 0.5$. The crack angle is $\beta \in [0, \pi/2]$. Poisson’s ratio and Young’s modulus are chosen as in Section 6.1.4. The analytical solutions for $k_1$ and $k_2$ for an infinite plate under plane strain conditions are

$$ k_1 = q_y \cdot \sqrt{a\pi} \cdot \cos^2 \beta \quad k_2 = q_y \cdot \sqrt{a\pi} \cdot \sin \beta \cos \beta, $$

(45)

cf. [36].

\[ \begin{array}{c}
\text{(a)} \\
\end{array} \]

Figure 17. (a) Sketch of the test case with an angled crack, (b) and (c) show $k_1$ and $k_2$ for different angles $\beta$, respectively.

Results are obtained on meshes with $n_{d}^{el} = \{20, 40, 60, 80, 100\}$ elements in each dimension. The local refinement is realized on both sides of the crack. Only the approach with DOFs at the hanging nodes is considered as constrained approximation yields similar results. The convergence to $k_1$ and $k_2$ is shown in Figures 17(b) and (c). Excellent agreement with the exact values is obtained for the entire range of $\beta$. The results are only shown for 20 and 100 elements per dimension and are compared to the results of Liu et al. [36] obtained on a uniform $100 \times 100$ mesh with a special version of the branch enrichment. It is found that already the refined $20 \times 20$ mesh with hanging nodes obtains results that are more accurate than that of [36].
6.2. Applications in fluid mechanics

6.2.1. Governing equations of incompressible flows

The domain Ω contains two different, immiscible incompressible Newtonian fluids in Ω₁ and Ω₂, respectively, so that Ω = Ω₁ ∪ Ω₂. Throughout this work, Ω is considered a time-independent closed container, whereas Ω₁(t) and Ω₂(t) change in time. The (moving) interface between the two fluids is denoted Γ_d. The normal vector on Γ_d is called \( \hat{n} \) and points from Ω₁ to Ω₂. See Figure 18 for a sketch of the situation. The boundary Γ = ∂Ω is decomposed into two disjoint parts \( \partial Ω = Γ_u ∪ Γ_h \).

The governing equations are now given in strong form, see e.g. [31, 53, 21]. Let \( u(x, t) \) be the velocities and \( p(x, t) \) the pressure; \( ϱ_i \) and \( μ_i \), with \( i = (1, 2) \) are the density and dynamic viscosity of the two fluids, respectively; \( f \) are volumetric forces such as gravity. The fluids inside \( Ω_i × (0, t_{\text{end}}) \), \( i = (1, 2) \), are modeled by the instationary, incompressible Navier-Stokes equations in velocity-pressure formulation

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nabla \cdot σ &= ϱ_i f, \\
\nabla \cdot u &= 0.
\end{align*}
\]  

The stress tensor \( σ \) of the Newtonian fluids is given as

\[
σ(u, p) = -pI + 2μ \varepsilon(u), \quad \text{with } \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),
\]

where \( I \) is the identity tensor. Dirichlet and Neumann boundary conditions on the outer boundary of \( Ω \) are

\[
\begin{align*}
u &= \hat{u} \quad \text{on } Γ_u × (0, t_{\text{end}}), \\
σ \cdot n &= \hat{h} \quad \text{on } Γ_h × (0, t_{\text{end}}),
\end{align*}
\]

where \( \hat{u} \) and \( \hat{h} \) are prescribed velocities and stresses. The following conditions typically apply...
at the interface
\begin{align}
[u]_{\Gamma_d} &= 0 \quad \text{on } \Gamma_d \times (0, t_{\text{end}}), \\
-\sigma_{\Gamma_d} \cdot \mathbf{n} &= \gamma \cdot \kappa \cdot \mathbf{n} \quad \text{on } \Gamma_d \times (0, t_{\text{end}}).
\end{align}
(51)
Here, $[f]_{\Gamma_d}$ is the jump of property $f$ across the interface $\Gamma_d$, $\gamma$ is the surface tension coefficient (material parameter) and $\kappa$ is the curvature of $\Gamma_d$. As an initial condition, a divergence-free velocity field $\mathbf{u}_0$ is specified over $\Omega$,
\begin{equation}
\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega \text{ at } t = 0.
\end{equation}
(53)

Again, we omit the weak form of this problem for brevity, see e.g. [28]. It is noted that stabilization is needed which is realized by a Petrov-Galerkin weak form. In particular, the streamline-upwind Petrov-Galerkin (SUPG) and pressure-stabilizing Petrov-Galerkin (PSPG) formulation are employed [55, 56, 13].

\subsection{6.2.2. Sloshing tank}
A two-fluid flow in a sloshing tank is considered. The setup of the test case follows the description in [34, 28] and is depicted in Figure 19. The domain $\Gamma$ is a container with height 1.5 m and width 1.0 m. The two fluids in $\Omega_1$ and $\Omega_2$ are separated by a sinusoidal interface $\Gamma_d = \{(x, y) : y = 1.01 + 0.1 \cdot \sin((x - 0.5) \cdot \pi), \ 0 \leq x \leq 1\}$, see Figure 19(a). The density of the fluids in $\Omega_1$ and $\Omega_2$ is $\rho_1 = 1000$ kg/m$^3$ and $\rho_2 = 1$ kg/m$^3$, respectively, and the viscosity is $\mu_1 = 1$ kg/s/m and $\mu_2 = 0.01$ kg/s/m. No surface tension is considered here. A volume force (gravitation) of $f_y = -g = -1.0$ m/s$^2$ is considered. Slip-conditions are assumed along the walls of the tank, and $p = 0$ N/m$^2$ is set along the upper boundary. The situation is observed for $t = (0s, 20s)$.

For this test case, we have used a coarse mesh of $4 \times 6$ elements (before the refinement). Figure 20 compares the meshes with corresponding interface positions for different refinement levels for the sloshing tank test case at $t = 20s$. Since the interface position moves with time, we have employed the criterion in equation (3) where all elements within a certain distance $d$ from the interface are refined. This is crucial since in order to maintain the desired level of accuracy, the interface position should always fall within the finest refinement region. For this test case, we have chosen $d = 0.05m$. We now perform a closer examination of the differences between the two approaches, i.e., hanging nodes with DOFs and hanging nodes without DOFs. In the figures, we shall employ the abbreviation “HN with DOFs” to depict the first approach and the abbreviation “HN w/o DOFs” to refer to the second approach. Figure 21(a) shows the variation of the water level on the left side of the sloshing tank with time for the two approaches at refinement levels 1 and 5. It can be seen that due to the viscous damping, the amplitude of the height decreases. The frequency of the oscillation is 0.279 Hz as shown in the reference solutions given in [34, 28]. As can be observed, differences in the results are negligible between the two approaches. However, as expected, a significant difference exists between a level 1 refinement and a level 5 refinement. Figure 21(b) compares the variation of water level with time for the approach with DOFs at the hanging nodes for refinement levels 1 to 5. As
can be seen, the height converges to a certain position as the refinement level increases. Figure 21(c) makes the same comparison for the constrained approximation. We do not observe any significant deviations from the conclusion drawn earlier for the approach with DOFs at the hanging nodes.

Finally, we perform a study on the mass conservation properties of the two different approaches. Since both fluids are immiscible and incompressible, the areas of the two fluids should be conserved over time. The domain $\Omega_1$ has the initial area of $A = 1.0 \cdot 1.01 = 1.01m^2$. It is conceivable that the higher the resolution of the mesh in the vicinity of the interface, the better is the area conservation. This is because the position of the interface is interpolated from the level-set function whose accuracy depends directly on the mesh resolution. Figure 22(a).
Figure 21. Comparison of the water level on the left side of the sloshing tank; (a) compares both approaches for the hanging nodes for refinement levels 1 and 5, (b) compares refinement levels 1 to 5 for DOFs at the hanging nodes, and (c) compares refinement levels 1 to 5 for the constrained approximation.

compares the development of the area $\Omega_1$ over time for the two approaches for refinement levels 1 and 5. Only a small difference can be found between the two approaches and this difference becomes almost indiscernible as the refinement level increases from 1 to 5. In other words, as the refinement level increases, the difference in results between the two approaches becomes more negligible. Figure 22(b) compares the area conservation for refinement levels 1 to 5 for the approach with DOFs at the hanging nodes. We observe a clear convergence to the exact area of 1.01$\text{m}^2$ as the refinement level increases. We omit the result for the constrained approximation at it looks almost identical.

6.2.3. Rising bubble For the bubble test case, the domain is sketched in Figure 23(a). We have chosen an initial bubble radius of $r = 5 \cdot 10^{-6}\text{m}$. Surface tensions effects are considered in this test case leading to a jump in the pressure field. Consequently, the pressure field is enriched with the step-enrichment (8). The surface tension coefficient is specified as $\gamma = 10^{-3}\text{kg/s}^2$ and the densities of the fluids are chosen so that the Eötvös number results as $Eo = 1000$. The situation is observed until $t_{\text{end}} = 25\text{s}$ and 1000 time steps are used. The evolution of the interface in time is shown in Figure 23(b). More details on this test case, including reference
Figure 22. Comparison of mass conservation for the sloshing tank test case. (a) compares both approaches for refinement levels 1 and 5. (b) compares refinement levels 1 to 5 for the approach with DOFs at the hanging nodes.

solutions, are found in [34, 28]. Here again, we use the criterion in equation (3) where all elements within a certain distance $d$ from the interface are refined and $d = 0.1 \cdot r$. A coarse mesh of $6 \times 12$ elements has been used before the refinement.

Figure 23. (a) Problem statement for the rising bubble test case, (b) development of the interface over time.

Figure 24 shows the shapes of the bubble at the final time $t_{\text{end}}$ for four levels of refinement ($n_{\text{ref}} = 2$ to $n_{\text{ref}} = 5$). Results are shown only for the approach with DOFs at the hanging nodes since the constrained approximation yields almost identical results and differences are
not discernible by visual inspection. As can be observed, due to topological changes (i.e. the pinching apart of the bubble), the position of the interface cannot be reasonably resolved on a coarse mesh with a low refinement level. In this case, higher levels of refinement (levels 3 and above) are required in order to resolve the interface position to a reasonable accuracy. The advantage of adaptive mesh refinement in the vicinity of the interface is apparent in this case since for the same resolution of the interface, a uniform mesh refinement will require a much greater computational effort.

![Comparison of the shapes of the bubble at different refinement levels at t = 25s.](image)

7. Conclusions

Adaptive mesh refinement with hanging nodes via two different approaches in the context of the XFEM has been investigated in this paper. The first approach assumes that DOFs are present at hanging nodes and special shape functions are constructed for both the hanging and regular nodes satisfying both conformity and the partition-of-unity property. A consequence of this approach is that these special shape functions possess kinks which necessitate the use of a special quadrature taking into account the presence of such kinks in addition to the discontinuities already possessed by the enrichment function. An important advantage of this approach is that the choice of the enriched nodes is intuitive and is consistent with the use of the XFEM on conforming meshes.

The second approach assumes that no DOFs exist at the hanging nodes by constraining these DOFs to be the average of the DOFs at neighbouring nodes. Also known as the constrained approximation, non-conforming bi-linear shape functions are constructed for all the nodes and the system matrix is assembled in the usual manner. Thereafter, one may adopt two implementationally different strategies which lead to identical results. The first strategy enforces the constraints on the hanging node DOFs via a Lagrange multiplier approach. In the second strategy, the system matrix is multiplied by a connectivity matrix, which removes DOFs associated with the hanging nodes and constrains these DOFs to the neighboring nodes.
Although fewer DOFs result from this second approach, the choice of the enriched nodes is not as apparent as in the first approach. However, a special viewpoint is elucidated in this study where the constraints are applied to the non-conforming bilinear shape functions, resulting in a set of conforming shape functions on the 1-irregular mesh. These shape functions only exist for the regular nodes and satisfy the partition-of-unity property in the domain. It is then clear that all (regular) nodes whose supports are crossed by the discontinuity are to be enriched.

The numerical results achieved in applications from solid and fluid mechanics are summarized as follows: Both approaches lead to very similar results. Often, the results for the approach with DOFs at the hanging nodes performs slightly better (by some constant) due to the larger numbers of DOFs compared to constrained approximation. Furthermore, it is found that heuristic local refinements near the interface have the potential to improve the results drastically. This is due to the improved resolution of the interface position and the ability to capture steep gradients near the interface.

The numerical results have demonstrated that the powerful combination of adaptive mesh refinement and an enrichment of the approximation space (XFEM) can handle a large class of problems with moving interfaces/fronts while producing highly accurate results and yet incurring only a fraction of the computational cost associated with uniform mesh refinements.

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