SUPG stabilization for convection-dominated problems with Meshfree Methods

Thomas Fries
Prof. Hermann G. Matthies

March 25, 2003
1) Introduction

2) Model problem

3) Stabilization of the Navier-Stokes equations

4) Results

5) Conclusion
1) Introduction

- Convection-dominated problems in Eulerian formulation require stabilization.
- SUPG: "streamline upwind Petrov-Galerkin".
- Basic idea: introduce the right amount of artificial diffusion in streamline direction only.
- Realized by a modification of the weak form

\[ \int w(PDE) = 0 \quad \rightarrow \quad \int \left( w + \tau \frac{\partial w}{\partial x_i} \right)(PDE) = 0 \]

- \( \tau \) is called stabilization parameter.
2) Model problem

model problem: 1D advection-diffusion equation

\[ u \frac{\partial \varphi^T}{\partial x} - K \frac{\partial^2 \varphi^T}{\partial x^2} = 0 \]

\[ \int_{\Omega} \left( w + \tau \ u \frac{\partial w}{\partial x} \right) \left( u \frac{\partial N^T}{\partial x} - K \frac{\partial^2 N^T}{\partial x^2} \right) d\Omega \ \varphi = 0 \]

in Finite Element Method (FEM):
- element stabilization is standard
- one \( \tau_e \) for each element

in Meshfree Methods (MMs):
- nodal stabilization
- one \( \tau_i \) for each global equation
2) Model problem

nodal stabilization

\[
\int_{\Omega_i} \left( w_I + \tau_I u \frac{\partial w_I}{\partial x} \right) \left( u \frac{\partial N^T}{\partial x} - K \frac{\partial^2 N^T}{\partial x^2} \right) d\Omega \; \varphi = 0
\]

\[
\tau_I = - \frac{\int_{\Omega_i} \left( w_I \right) \left( u \frac{\partial N^T}{\partial x} - K \frac{\partial^2 N^T}{\partial x^2} \right) d\Omega \; \varphi_{\text{ex}}^i}{\int_{\Omega_i} \left( u \frac{\partial w_I}{\partial x} \right) \left( u \frac{\partial N^T}{\partial x} - K \frac{\partial^2 N^T}{\partial x^2} \right) d\Omega \; \varphi_{\text{ex}}^i}
\]

with \( \varphi_{\text{ex}}^i = c_1 e^{\frac{u}{K} x} + c_2 \)

- **local** criteria,
- only valid for shape functions with Kronecker-delta property
- e.g. for linear FEM follows:

\[
\tau_I = \frac{\Delta x}{2u} \left( \coth(Pe) - \frac{1}{Pe} \right)
\]
2) Model problem

Problem for MMs:
Meshfree shape functions do not have Kronecker-delta property in general.

\[ \varphi(x) = N^T(x) \varphi \]

evaluated at nodes \( x \):

\[ \varphi(x) = N^T(x) \varphi \]

\[ \varphi^{real} = D \varphi^{fict} \]

It follows:

\[ \varphi^{ex, fict} = D^{-1} \varphi^{ex} \]

\[ \tau_l = - \frac{\int_{\Omega_l} \left[ w_I \frac{\partial N^T}{\partial x} \right] d\Omega \ D^{-1} \varphi^{ex}}{u \int_{\Omega_l} \left[ \frac{\partial w_I}{\partial x} \frac{\partial N^T}{\partial x} \right] d\Omega \ D^{-1} \varphi^{ex}} \]

- global criteria, as \( D^{-1} \) is a full matrix.
2) Model problem

original, local, no $\delta_{ij}$-property

transformed, global, $\delta_{ij}$-property
2) Model problem

\[ \tau_i = - \frac{\int_{\Omega_i} \left[ w_l \frac{\partial N^T}{\partial x} \right] d\Omega \quad D^{-1} \quad \phi^{ex} \quad \text{large} \frac{u}{K}}{u \int_{\Omega_i} \left[ \frac{\partial w_l}{\partial x} \frac{\partial N^T}{\partial x} \right] d\Omega \quad D^{-1} \quad \phi^{ex}} \]

Conclusions from the 1D analysis:

- The matrix \( D^{-1} \) must be taken into account.
- For large \( u/K \) stabilization relies on the global downstream node.
- Any stabilization based on local parameters cannot be well suited.
- For moderate \( u/K \) the choice of a „reduced good“ matrix \( D^{-1} \) may be justified.
- MMs with Kronecker-delta property are advantageous for stabilization, because then \( D = D^{-1} = I \).
2) Model problem

model problem: 2D advection-diffusion equation

\[
\int_{\Omega} \left[ w + \tau \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} \right) \right] \left( u \frac{\partial N^T}{\partial x} + v \frac{\partial N^T}{\partial y} - K \frac{\partial^2 N^T}{\partial x^2} - K \frac{\partial^2 N^T}{\partial y^2} \right) d\Omega \varphi = 0
\]

\[
\tau_i = - \frac{\int_{\Omega_i} \left[ w_i \frac{\partial N^T}{\partial \xi} \right] d\Omega \, D^{-1} \varphi^{ex}}{\int_{\Omega_i} \left[ \frac{\partial w_i}{\partial \xi} \frac{\partial N^T}{\partial \xi} \right] d\Omega \, D^{-1} \varphi^{ex}} \quad \text{with} \quad \varphi^{ex} = e^{\frac{u_x + v_y}{K^+}} = e^{\frac{v_y}{K^+}}
\]
3) Stabilization of the NS equations

2D incompressible Navier-Stokes equations in linearized form

\[
\int w_i \left[ \rho \left( u_j^m \frac{\partial u_i^{m+1}}{\partial x_j} - f_i \right) - \frac{\partial \sigma_{ij}^{m+1}}{\partial x_j} \right] d\Omega = 0
\]

\[
\int q \frac{\partial u_i^{m+1}}{\partial x_i} = 0
\]

- Analogy with the advection-diffusion equation as long as
  \[ u_j^m \approx u = \text{const} \quad \text{and} \quad v_j^m \approx v = \text{const} \]
  which is true for sufficiently small \( \Omega \) (can be controlled with the dilatation parameter).
- This justifies usage of the same stabilization parameter.
4) Results: 2D driven cavity

- SUPG stabilization for convection-dominated problems with Meshfree Methods -

Re=1000

\( l_x = 1.0 \)

\( l_y = 1.0 \)

\( u = 1.0 \)

Meshfree shape functions without \( \delta_{ij} \)-property and neglecting \( D^{-1} \)

Meshfree shape functions with \( \delta_{ij} \)-property, \( D^{-1} = I \)

no stabilization:
- oscillations
- no convergence
5) Conclusion

- Nodal stabilization is required for MMs. The stabilization parameter can be defined as
  \[\tau_i = -\frac{\int_{\Omega_i} \left[w_i \frac{\partial N^T}{\partial \xi}\right] d\Omega \ D^{-1} \phi_{ex}}{\nu \int_{\Omega_i} \left[\frac{\partial w_i}{\partial \xi} \frac{\partial N^T}{\partial \xi}\right] d\Omega \ D^{-1} \phi_{ex}}\]

- Meshfree shape functions without Kronecker-delta property
  - require solving systems of equations (D^{-1})
  - lead to non-local stabilization criterium
- Meshfree shape functions with Kronecker-delta property
  - lead to local stabilization criterium
  - give good results in fluid dynamics (if \(\Omega_i\) is sufficiently small)