Simulation of Incompressible Flows with Coupled FEM/EFG Shape Functions

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Contents

- Motivation: In complex flow problems, a mesh may be difficult to maintain. Then, meshbased methods like the FEM fail.

- Meshfree Methods
- Stabilization
- Coupling
- Numerical Results
- Summary and Outlook
Meshfree Methods

Meshfree Methods in Fluid Mechanics:
- Often collocation methods instead of Galerkin methods
  
<table>
<thead>
<tr>
<th></th>
<th>Collocation</th>
<th>Galerkin</th>
</tr>
</thead>
<tbody>
<tr>
<td>calculation time</td>
<td>fast</td>
<td>slow</td>
</tr>
<tr>
<td>accuracy</td>
<td>low</td>
<td>high</td>
</tr>
</tbody>
</table>

- Lagrangian viewpoint is standard, Eulerian viewpoint is rarely used
  
<table>
<thead>
<tr>
<th></th>
<th>Lagrange</th>
<th>Euler (ALE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>stabilization needed</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>particle clustering</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>boundary conditions</td>
<td>difficult</td>
<td>easy</td>
</tr>
<tr>
<td>coupling with FEM</td>
<td>difficult</td>
<td>easy</td>
</tr>
</tbody>
</table>
Meshfree Methods

- Chosen meshfree method:
  - Eulerian or ALE formulation,
  - Galerkin setting,
  - Based on moving least-squares (MLS) shape functions.
  Closely related to the element-free Galerkin (EFG) method.

- For the success this method,
  - stabilization and
  - coupling with meshbased methods
  is needed.
Meshfree Methods

Moving Least Squares (MLS)

- Local approximation around an arbitrary point
  \[ u^h(x) = p^T(x)a(x) \]
  - \( p^T(x) \): complete basis vector
    (depending on dimension and consistency order)
  - \( a(x) \): unknown coefficients of the approximation
- Minimize a weighted error functional
  \[ J(a) = \sum_{i=1}^{N} w(x - x_i) \left[ p^T(x)a(x) - u(x_i) \right]^2 \]
  - \( w(x - x_i) \): weighting function (locality)
  - Results in a system of equation for the coefficients
Meshfree Methods

- For the approximation follows:

\[ u^h(x) = \left( p^T(x) \right) \left( M^{-1}(x) B(x) \right) u \]

\[ N^T(x) : \text{meshfree shape functions} \]

sufficient overlap of the supports required

---

MLS shape functions

FEM shape functions
Stabilization

- Stabilization is necessary for
  - Advections terms: NS-eqts. in Eulerian or ALE formulation
  - „Equal-order-interpolation“: incompr. NS-eqts. with the same shape functions for velocity and pressure.

- Stabilization methods:
  - SUPG/PSPG
  - GLS
  - Others: FIC, upwinding, test-function shift; pressure projection, divergence-free interpolation space; discontinuity capturing
Stabilization

- Incompressible Navier-Stokes equations:
  \[
  \rho \left( u_{,t} + u \cdot \nabla u - f \right) - \nabla \cdot \sigma = 0
  \]
  \[
  \nabla \cdot u = 0
  \]
  with \( \sigma(u, p) = -pI + 2\mu \varepsilon(u), \quad \varepsilon(u) = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right) \)

- Unstabilized weak form (Bubnov-Galerkin):
  \[
  \int \tilde{w} \rho \left( \tilde{u}_{,t} + \tilde{u} \cdot \nabla \tilde{u} - \tilde{f} \right) d\Omega + \int \varepsilon(\tilde{w}) : \tilde{\sigma}(\tilde{u}, \tilde{p}) \ d\Omega + \\
  + \int \tilde{q} \nabla \cdot \tilde{u} \ d\Omega = \oint \tilde{w} \tilde{h} \ d\Gamma
  \]
Stabilization

- SUPG/PSPG-stabilized weak form:
  \[ ... + \sum_{i=1}^{n_{el}} \int_{\Sigma} \left( \tilde{u} \cdot \nabla \tilde{w} + \frac{1}{\rho} \nabla \tilde{q} \right) + \text{SUPG/PSPG} \]

\[
\left[ \rho \left( \tilde{u}_t + \tilde{u} \cdot \nabla \tilde{u} - f \right) - \nabla \cdot \tilde{\sigma}(\tilde{u}, \tilde{p}) \right] d\Omega + ...
\]

- GLS-stabilized weak form:
  \[ ... + \sum_{i=1}^{n_{el}} \int_{\Sigma} \frac{1}{\rho} \left( \rho \tilde{u} \cdot \nabla \tilde{w} - \nabla \cdot \tilde{\sigma}(\tilde{w}, \tilde{q}) \right) + \text{GLS} \]

\[
\left[ \rho \left( \tilde{u}_t + \tilde{u} \cdot \nabla \tilde{u} - f \right) - \nabla \cdot \tilde{\sigma}(\tilde{u}, \tilde{p}) \right] d\Omega + ...
\]
Stabilization

- The structure of the stabilization methods may be applied to meshbased and meshfree methods.
- The stabilization parameter $\tau$ requires special attention:
  - Global criterium (depending on the global downstream-node) for $N_i(x_j) \neq \delta_{ij}$
  - Local formulas like the meshbased standard formulas are only valid for small supports.
- SUPG/PSPG-stabilization turned out to be slightly less diffusive than GLS.
Coupling

- Coupling has always the aim to combine certain advantages from meshbased and meshfree methods.
- Advantages of meshbased methods:
  - Efficiency
- Advantages of meshfree methods:
  - Continuity: RKEM
  - Enrichment: PUM, PUFEM, GFEM, XFEM
  - No mesh: coupled meshfree/meshbased shape fcts.
  - Adaptivity
Coupling

- Our aim: a fluid solver for complex geometrical problems which allows
  - Large deformation of the domain
  - Moving and rotating objects

  MMs shall be used where a mesh causes problems, in the rest of the domain we use efficient meshbased standard methods (FEM).

- The coupling is realized on shape function level.

meshbased area

transition area

meshfree area
Coupling

- Coupling approaches:
  - Huerta et al.: Coupling with modified MLS technique, which considers the FEM influence in the transition area.

\[
N_i(x) = \left[ p^T(x) - \sum_{j \in I^{\text{FEM}}} N_j^{\text{FEM}}(x) p^T(x_j) \right]
\]

\[
M^{-1}(x) w(x - x_i) p(x_i)
\]

- Belytschko et al.: Coupling with a ramp function.

\[
N_i(x) = \left[ 1 - R(x) \right] N_i^{\text{FEM}} + R(x) N_i^{\text{MLS}}
\]

- Modifications are required in order to obtain coupled shape functions that may be stabilized reliably.
Coupling

- Original approach of Huerta et al.

- Modification: Add MLS nodes along $\Omega^{\text{FEM}} \cap \Omega^{\ast}$ and superimpose shape functions there.

$\rho_i > 2.0\Delta x$

$\rho_i > 1.0\Delta x$
Coupling

- Original approach of Belytschko et al.

\[ N_i = \left[ 1 - R(x) \right] N_i^{\text{FEM}} + R(x) N_i^{\text{MLS}} \]

- Modification: Restrict MLS nodes to \( \Omega^{\text{MLS}} \cup \Omega^* \).
Numerical Results (validation)

- Driven cavity (reference: Ghia)
Numerical Results (validation)

- Flow around a cylinder (reference: Turek)
Numerical Results

- Rotating object

\[ u_{\text{max}} = 1.5, \quad v = 0 \]

\[ u=0, \ v=0 \]

\[ \text{Re} = 100 \]
Numerical Results

- Moving flap

![Diagram showing numerical results with annotations and labels](image)

- $u_{\text{max}} = 1.5$
- $v = 0$
- $\alpha$
- $\Omega^\text{FEM}$
- $\Omega^\text{MLS}$
- $u = 0, v = 0$
- $\text{Re} = 100$

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Thomas Fries, Coupled FEM/EFG Simulation

Slide: 19
Summary and Outlook

- **Summary**
  - Meshfree Method: Eulerian (ALE) formulation, Galerkin setting, MLS-based $\Rightarrow$ EFG
  - Stabilization: SUPG/PSPG and GLS
  - Coupling: Modified approaches of Huerta and Belytschko
  - Numerical results of the coupled flow solver

- **Outlook**
  - More test cases
  - Coupled shape functions of higher order
  - Extension to three-dimensional problems