The Extended Finite Element Method for Boundary Layers

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Research group: “Numerical methods for discontinuities“

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Overview

- Motivation
- XFEM for Boundary Layers in 1D
- XFEM for Boundary Layers in 2D
- Conclusions & Outlook
Boundary Layers:

- occur in advection-diffusion problems such as flows.
- are characterised by high gradients normal to the wall.
Motivation

Numerical methods for Boundary Layers:

- **FEM**
  - refined mesh near the wall
  ⇒ many new elements ⇒ new DoF

- **XFEM**
  - adjust the approximation space,
    but does not manipulate the mesh

Find appropriate enrichment functions for BL
XFEM for BL in 1D

- Model problem: Advection-diffusion problem
  \[ c \cdot u_x - k \cdot u_{xx} = 0 \quad u(0) = 0, u(1) = 1 \]
  advection diffusion

- The exact solution has the form:
  \[ u = f \left( e^{c/k \cdot x} \right) \]
XFEM for BL in 1D

- The standard XFEM approximation:
  \[ u^h = \sum N_i u_i + \sum N_i \psi_1 a_i + \sum N_i \psi_2 b_i + \ldots \]
  enrichment 1  enrichment 2

- Enrichment functions are sought such that for all \( c/k \) ratios:
  - high accuracy is obtained
  - no oscillations occur
  - the approximation space remains sufficiently linearly independent

- Enrichment functions should be defined locally near the wall and independent of \( c \) and \( k \) coefficients
XFEM for BL in 1D

- This family of enrichment functions is found useful:

\[ \psi (x) = \frac{e^{n \cdot s_L(x)} - 1}{e^n - 1}, \]

where \( n \) scales the gradient and

\[ s_L(x) = \begin{cases} 
1, & \text{at the wall} \\
0 < s_L < 1, & \text{within } L \text{ layers from the wall} \\
0, & \text{else} 
\end{cases} \]

- The pair \((n, L)\) defines each \(\psi_i\)
Using only one enrichment function is not sufficient to cover the whole range of $c/k$ ratios without oscillations.

The "optimal" set of enrichment functions under the chosen criteria is:

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$n$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_1$</td>
<td>50</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_3$</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>$\psi_4$</td>
<td>10</td>
<td>3</td>
</tr>
</tbody>
</table>
XFEM for BL in 1D

L2 Norm for XFEM and FEM approximations

L2Norm

1.e-1

1.e-6

10

640 elements

Alaskar Alizada  The XFEM for Boundary Layers  Slide: 9
**XFEM for BL in 1D**

Max. oscillations of XFEM and FEM approximations

No oscillations for XFEM occur

- FEM, 10 elements
- FEM, 40 elements
- FEM, 160 elements

```
0
-0.5
-1.0
-1.5
-2.0
-2.5
```

```
1
10
100
1000
500
```

-2.5
XFEM for BL in 1D

Summary

For the proposed set of 4 enrichment functions:

- High accuracy is obtained
- No oscillations occur
- The approximation space is linearly independent
- The enrichment functions are locally defined and are independent of the problem coefficients
**XFEM for BL in 2D**

- Model problem: Advection-diffusion problem
  \[ c_x \cdot u,_{x} + c_y \cdot u,_{y} - k \cdot u,_{xx} - k \cdot u,_{yy} = 0 \]
  - advection
  - diffusion

- The exact solution has the form: \[ u = f(e^{c_x/k \cdot x+c_y/k \cdot y}) \]
XFEM for BL in 2D

- A similar set of enrichment functions is taken as for 1D:

\[ \psi (x) = \frac{e^{n \cdot s_L(x)} - 1}{e^n - 1}, \]

where \( n \) scales the gradient and \( s_L(x) \):

- is constructed by FE-shape functions
- is characteristic for the wall-distance
The 1D set of enrichment functions is straightforward extended to 2D with the same pairs $(\psi, \phi)$:

- $\psi_1$ with $n = 50$ and $L = 1$
- $\psi_2$ with $n = 15$ and $L = 1$
- $\psi_3$ with $n = 12$ and $L = 2$
- $\psi_4$ with $n = 10$ and $L = 3$
L2 Norm for XFEM and FEM approximations ($c_x = c_y = c$)
XFEM for BL in 2D

Max. oscillations of XFEM and FEM approximations

No oscillations for XFEM occur

FEM, 30 elements
FEM, 20 elements
FEM, 10 elements
Conclusions

A set of enrichment functions for XFEM is proposed that:

- captures arbitrary high gradients \textit{normal} to the wall
- captures moderate changes of the solution \textit{tangential} to the wall
- captures the properties of boundary layers without increasing the DoF´s significantly

Outlook

Using proposed enrichment functions in the XFEM, no mesh refinement for boundary layers is required.

This idea is going to be extended to fluid problems, starting with driven cavity at high Reynolds-numbers.
THANK YOU FOR YOUR ATTENTION!