Time Integration in the XFEM

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Model problem

- Instationary diffusion equation in 1d:
  \[ u,_{t} - k \cdot u,_{xx} = 0 \]

  with \[ k = \begin{cases} 
  k_1 & \text{for } x \leq x^*(t) \\
  k_2 & \text{for } x > x^*(t) 
\end{cases} \]

- The discontinuity \( x^*(t) \) moves in time.

\[ t = 1 \]

In space-time domain:
The exact solution has a kink at $x^*(t)$:
- The higher $k_1/k_2$, the stronger is the kink.
- The bigger $(x_B - x_A)$, the faster moves the discontinuity.
A review of space-time XFEM

- Discontinuous-Galerkin space-time XFEM used e.g. by [Chessa, IJNME, 2004], [Zilian, IJNME, 2008].

![Diagram showing space-time elements and weak form](image)

- Weak form for space-time:

\[
\int_{Q_n} w \ u_{,t} \ dQ + \int_{Q_n} k \cdot w_{,x} u_{,x} \ dQ + \int_{\Omega_n} w(t_n^+) [u(t_n^+) - u(t_n^-)] d\Omega = 0
\]
A review of space-time XFEM

- XFEM-approximation for $u(x, t)$:

$$u(x, t) = \sum N_i(x, t)u_i + \sum N_i(x, t)\Psi(x, t)a_i$$

- Note:
  - Enrichment is space-time dependent.
  - Cut space-time elements need special int.

- Properties of space-time XFEM:
  - Optimal convergence.
  - Rather time-consuming (number of DOFs is high, special integration in cut space-time elements).
Time-stepping in the XFEM

- Crank-Nicolson scheme: \( u_{t} = \frac{u^{n+1} - u^{n}}{\Delta t} \)

\[
u = \theta \cdot u^{n+1} + (1 - \theta) \cdot u^{n} \quad \text{with} \quad \theta = 0.5
\]

- Weak form for time-stepping:

\[
\frac{1}{\Delta t} \int w \, u^{n+1} - \frac{1}{\Delta t} \int w \, u^{n} + \theta \int k \, w_{,x} \, u^{n+1}_{,x} + (1 - \theta) \int k \, w_{,x} \, u^{n}_{,x} = 0
\]
Time-stepping in the XFEM

- XFEM-approximation (choice of shape functions):
  \[ u^{n+1} = \sum N_i(x)u_i + \sum N_i(x)\Psi(x, t_{n+1})a_i \]
  \[ u^n = \sum N_i(x)u_i + \sum N_i(x)\Psi(x, t_n)a_i \]
  Strd. FEM
  enrichment

- How to choose test functions \( w \)?
  [Chessa, IJNME, 2004: „Ambiguity in the choice of test functions“]
  \[ w^{n+1} = [N_i, N_i\Psi(x, t_{n+1})] \] crucial for regularity
  \[ w^\theta = [N_i, N_i\Psi(x, \theta \cdot t_{n+1} + (1 - \theta) \cdot t_n)] \]
  \[ w^n = [N_i, N_i\Psi(x, t_n)] \]
Time-stepping in the XFEM

- "Mixed" terms appear, e.g.:

\[ \frac{1}{\Delta t} \int_{\Omega} w^{n+1} \ u^n \ d\Omega \]

- \(w^{n+1}\) and \(u^n\) have inner-element kinks due to the enrichment.
  - The kink in \(w^{n+1}\) is based on \(x^*(t_{n+1})\).
  - The kink in \(u^n\) is based on \(x^*(t_n)\).
Time-stepping in the XFEM

- "Mixed" terms require new integration procedure in cut elements.
- Subdivide element w.r.t. to the discontinuities at $t_{n+1}$ and $t_n$:

1 element subdivided into 3 sub-elements for integration

integration point
Time-stepping in the XFEM

Convergence study: Influence of $k_1/k_2$.

\[ n \cdot k_1 = k_2 \]
Time-stepping in the XFEM

Convergence study: Influence of discontinuity speed $v^* = x^*_{,t}$.

$x^*(t) = x_A + t \cdot (x_B - x_A)$

$x_B = 1 - x_A$

$k_1 = 0.00025$, $k_2 = 0.00500$

$x_A = 0.2$

$x_A = 0.3$

$x_A = 0.4 - 10^{-6}$

$x_A = 0.45$
Extension to 2d

The same key ingredients:

- Test functions are chosen at \( t_{n+1} \).
- Subdivide element w.r.t. to the discontinuities at \( t_{n+1} \) and \( t_n \):

\[
\begin{align*}
&\text{integration point} \\
\end{align*}
\]
Numerical results: sloshing tank

\[ y^* = 1.01 \]

\[ \rho_1 = 1000 \]
\[ \mu_1 = 1 \]

\[ \rho_2 = 1 \]
\[ \mu_2 = 0.1 \]

\[ H = 1.5 \]

\[ L = 1.0 \]
Numerical results: sloshing tank

Results are compared with standard FEM and interface-tracking: Excellent agreement.
Num. results: collapsing water column

\[ \rho_2 = 1 \]
\[ \mu_2 = 0.1 \]
\[ \rho_1 = 1000 \]
\[ \mu_1 = 1 \]
\[ H = 0.45 \]
\[ L = 0.584 \]
Num. results: collapsing water column
Numerical results: rising bubble

\[ \rho_1 = 1 \]
\[ \mu_1 = 0.01 \]

\[ \rho_2 = 0.1 \]
\[ \mu_2 = 0.01 \]

\[ d = 0.1 \]

\[ L = 0.2 \]

\[ H = 0.4 \]
Numerical results: rising bubble

vorticity

pressure
Summary

- Proposal of time-stepping in the XFEM.
- Key ingredients:
  - Specification of the test function based on $t_{n+1}$.
  - Subdivision of elements w.r.t. the discontinuities at $t_{n+1}$ and $t_n$.
- Properties:
  - Accuracy depends on the kink and the speed of the discontinuity.
  - 2nd order convergence rate.