Higher-Order XFEM for Arbitrary Weak Discontinuities

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Columbus, Ohio
July 17, 2009
Contents

- Motivation
- Interpolation of interface
- Sub-cell integration
- Higher-order XFEM formulations
  - Standard XFEM
  - Modified abs-enrichment (Möes et. al., 2003)
  - Corrected XFEM (Fries, 2007)
- Numerical results
- Conclusion
Motivation

- High-order XFEM: what has already been done?
  - Higher-order convergence rates for **strong and weak, straight discontinuities in 2-D** have been achieved by previous workers (e.g. Laborde *et. al.*, 2005; Legay *et. al.*, 2006).
  - Higher-order convergence rates for **strong, curved discontinuities in 2-D** have been achieved (e.g. Dréau *et. al.*, 2008).

- This study considers higher-order XFEM for **weak, curved discontinuities in 2-D**. Both quadratic and cubic approximations will be investigated.
Contents

✓ Motivation

- Interpolation of interface
- Sub-cell integration
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  - Standard XFEM
  - Modified abs-enrichment (Möes et. al., 2003)
  - Corrected XFEM (Fries, 2007)

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- Conclusion
Interpolation of interface

The position of the interface is defined by the level-set method:

- The level-set function $\phi(x)$ is a signed distance function which stores the shortest distance to the discontinuity.
- The zero-level of the level-set function is the discontinuity.
Interpolation of interface

- Interpolation using finite element shape functions.

\[ \phi^h(\mathbf{x}) = \sum_{i \in I} N_{i}^{\text{FEM}}(\mathbf{x}) \phi_i \]
Contents

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✓ Interpolation of interface
  ▪ Sub-cell integration
  ▪ Higher-order XFEM formulations
    ▪ Standard XFEM
    ▪ Modified abs-enrichment (Möes et. al., 2003)
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 ▪ Conclusion
Sub-cell integration

- In the XFEM, integration of element matrices needs to take into account different material properties across interfaces.
Sub-cell integration

- Projection of integration points
Sub-cell integration

- Projection of integration points

Always end up with **triangular** and **quadrilateral** sub-cells with at most one curved side.

Quadrilateral sub-cell

Triangular sub-cells
Sub-cell integration

- Projection of integration points

Reference elements
Sub-cell integration

- Projection of integration points

![Reference elements diagram]

7/14/2009 K.W. Cheng, Higher-Order XFEM
Sub-cell integration

- Objective: Locate the 4 points on the interface

\[ \phi^h(\xi, \eta) = \sum_{i \in I^e} N_i(\xi, \eta) \phi_i = 0 \]
Sub-cell integration

- Projection of integration points

Reference elements
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✓ Motivation
✓ Interpolation of interface
✓ Sub-cell integration
  ▪ Higher-order XFEM formulations
    ▪ Standard XFEM
    ▪ Modified abs-enrichment (Möes et. al., 2003)
    ▪ Corrected XFEM (Fries, 2007)
  ▪ Numerical results
  ▪ Conclusion
Standard XFEM formulation

- Standard XFEM approximation:

\[ u^h(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) u_i + \sum_{i \in I^*} N^*_i(\mathbf{x}) a_i \psi(\mathbf{x}) \]
Standard XFEM formulation

- Standard XFEM approximation:

\[ u^h(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) u_i + \sum_{i \in I^*} N_i^*(\mathbf{x}) a_i \psi(\mathbf{x}) \]

- For weak discontinuities

\[ \psi(\mathbf{x}) = \text{abs}(\phi(\mathbf{x})) \]
Problems in blending elements

- Using the **standard XFEM formulation with the abs-enrichment** leads to problems in blending elements (e.g. Sukumar *et. al.*, 2001; Chessa *et. al.*, 2003).

- **Remedies** for such problems
  - Modified abs-enrichment (Möes *et. al.*, 2003)
  - Corrected XFEM (Fries, 2007)
Problems in blending elements

- Partition-of-unity functions $N^*_i(x)$ do not build a partition-of-unity over the blending elements.
Problems in blending elements

- Introduces *unwanted parasitic terms* into the approximation space of the blending elements.
- Degrades both *accuracy* and *convergence* rates.
- Affects the *abs-enrichment*
Modified abs-enrichment (Möes et. al., 2003)

\[ \psi^{mod}(\mathbf{x}) = \sum N_i(\mathbf{x}) \abs(\phi_i) - \abs\left( \sum N_i(\mathbf{x}) \phi_i \right) \]
Modified abs-enrichment

- Modified abs-enrichment (Möes et. al., 2003)

\[
\psi_{\text{mod}} (\mathbf{x}) = \sum N_i (\mathbf{x}) \, \text{abs} (\phi_i) - \text{abs} \left( \sum N_i (\mathbf{x}) \phi_i \right)
\]
Modified abs-enrichment

- Modified abs-enrichment (Möes et. al., 2003)

\[ \psi^{\text{mod}}(\mathbf{x}) = \sum N_i(\mathbf{x}) \cdot \text{abs}(\phi_i) - \text{abs} \left( \sum N_i(\mathbf{x}) \phi_i \right) \]
Modified abs-enrichment (Möes et. al., 2003)

\[
\psi^{mod}(\mathbf{x}) = \sum N_i(\mathbf{x}) \, \text{abs}(\phi_i) - \text{abs}\left(\sum N_i(\mathbf{x}) \, \phi_i\right)
\]
Corrected XFEM formulation

- Corrected XFEM (Fries, 2008)

\[ u^h(x) = \sum_{i \in I} N_i(x) u_i + \sum_{i \in J^*} N_i^*(x) a_i \psi^{\text{mod}}(x) \]
Corrected XFEM formulation

- Corrected XFEM (Fries, 2008)

\[ R(\mathbf{x}) = \sum_{i \in I^*} N^*_i(\mathbf{x}) \]

\[ \psi^{\text{mod}}(\mathbf{x}) = \psi(\mathbf{x}) \cdot R(\mathbf{x}) \]
Contents

✓ Motivation
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✓ Sub-cell integration
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  ✓ Standard XFEM
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▪ Conclusion
Numerical results

- **Bi-material with a circular inclusion** (Sukumar et. al., 2001; Legay et. al., 2005; Fries, 2008)

![Diagram of bi-material with a circular inclusion](image)

**Material Properties**

\[
E_1 = 10, \nu_1 = 0.3 \\
E_2 = 1, \nu_2 = 0.25
\]

**Plane-strain conditions**

\[
u_r(b, \theta) = r \\
u_\theta(b, \theta) = 0
\]

Loading results from constant radial displacement

25, 1
32, 12
22, 11
35, 18
32, 22
35, 18

2009 K.W. Cheng, Higher-Order XFEM 28
Numerical results

- Computational domain

![Diagram of computational domain with reproducing and blending elements]
Numerical results

- Computational domain

![Diagram showing computational domain with Dirichlet boundary conditions: $u(0, \pm 1) = 0$ and $v(\pm 1, 0) = 0$. The domain is marked with reproducing and blending elements.]

Dirichlet boundary conditions:

- $u(0, \pm 1) = 0$
- $v(\pm 1, 0) = 0$
Numerical results

- Computational domain

![Diagram showing computational domain with Dirichlet and Neumann boundary conditions.]

**Dirichlet boundary conditions**

\[ u(0, \pm 1) = 0 \]
\[ v(\pm 1, 0) = 0 \]

**Neumann boundary conditions**

\[ \sigma_{xx} n_x + \sigma_{xy} n_y = t_x \]
\[ \sigma_{xy} n_x + \sigma_{yy} n_y = t_y \]

- Reproducing elements
- Blending elements
Numerical results

- Quadratic elements (i.e. order of $N_3(x) = 2$)

![Graph 1: Strd. XFEM vs Corr. XFEM](image1)

- ![Graph 2: Strd. XFEM vs Mod. abs-enr](image2)
Numerical results

- **Cubic elements** (i.e. order of \( N_{\delta} (x) = 3 \))

![Graphs showing L2 Norm vs Element Size for different methods and orders.](image)
Numerical results

- Comparison of convergence rates

<table>
<thead>
<tr>
<th></th>
<th>Quadratic</th>
<th>Cubic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Standard XFEM</strong></td>
<td>2.5/3.0</td>
<td>3.4/4.0</td>
</tr>
<tr>
<td>(order $N_i^*(\mathbf{x}) = 1$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Modified abs-enrichment</strong></td>
<td>2.4/3.0</td>
<td>2.5/3.0</td>
</tr>
<tr>
<td><strong>Corrected XFEM</strong></td>
<td>2.8/3.0</td>
<td>3.7/4.0</td>
</tr>
<tr>
<td>(order $N_i^*(\mathbf{x}) = \text{order } N_i(\mathbf{x})$)</td>
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</tr>
</tbody>
</table>

- Corrected XFEM has **higher accuracy** as well.
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  ✓ Standard XFEM
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Conclusion

- Standard XFEM yields best results when first order partition-of-unities $N_i^* (x)$ are used; however, convergence rates are still **suboptimal**.
- Modified abs-enrichment yields **suboptimal** convergence rates for all orders of $N_i^* (x)$.
- Corrected XFEM yields **close-to-optimal** convergence rates when order of $N_i (x) = \text{order of } N_i^* (x)$.
- Higher-order XFEM for **curved** weak discontinuities is possible.
Thank you!

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