The Extended Finite Element Method for Convection-Dominated Problems

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Outline

Enrichment functions for convection dominated problems
  Motivation & formulation
  Enrichment functions

Optimal set of enrichment functions

Numerical results

Conclusions
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    Motivation & formulation
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Conclusions
Standard FEM (No Stabilization)
Standard FEM (No Stabilization)
Stabilized FEM (SUPG Stabilization)
Stabilized FEM (SUPG Stabilization)
XFEM Formulation

- Instead of stabilization and/or refinement we want to enrich the approximation space.
Enrichment functions for convection dominated problems

**Enrichment functions**

**XFEM Formulation**

- Instead of stabilization and/or refinement we want to enrich the approximation space.
- XFEM approximation.
Enrichment functions for convection dominated problems

**XFEM Formulation**

- Instead of stabilization and/or refinement we want to enrich the approximation space.
- XFEM approximation.
  - Standard finite element approximation.

\[
u^h(x) = \sum_{i \in I} N_i(x) u_i + \sum_{j=1}^{\xi} \sum_{i \in I^\ast} N_{i^\ast}(x) \psi_j(x) a_{ij},\]

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Enrichment functions for convection dominated problems

Enrichment functions

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  - Standard finite element approximation.
  - Enrichment.

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 u^h(x) = \sum_{i \in I} N_i(x) u_i + \sum_{i \in I^*} N_i^*(x) \cdot \psi(x) a_i ,
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  - Standard finite element approximation.
  - Enrichment.

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Enrichment functions for convection dominated problems

**Enrichment functions**

**XFEM Formulation**

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Enrichment functions for convection dominated problems

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- Set of nodes whose support is cut by the interface.
Enrichment functions for convection dominated problems

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- Set of nodes whose support is cut by the interface.
- Partition-of-unity functions, usually but not necessarily, the same as \( N_i \).
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- Enrichment function.
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- Set of nodes whose support is cut by the interface.
- Partition-of-unity functions, usually but not necessarily, the same as \( N_i \).
- Enrichment function.
- Additional degrees of freedom.
• Enrichment Functions
Enrichment functions for convection dominated problems

**Enrichment functions**

- Enrichment Functions
  - Weak discontinuity $\rightarrow$ Abs-Enrichment
Enrichment functions for convection dominated problems

- Enrichment Functions
  - Weak discontinuity $\rightarrow$ Abs-Enrichment
  - Strong discontinuity $\rightarrow$ Sign/Heaviside-Enrichment
• High gradient enrichment function [Patzák & Jirásek, 2003].

\[
\psi(\phi, \epsilon) = \begin{cases} 
0, & \text{if } \phi < -\epsilon, \\
\frac{315}{256\epsilon} \int_{-\epsilon}^{\phi} \left(1 - \frac{\xi^2}{\epsilon^2}\right)^4 d\xi, & \text{if } |\phi| \leq \epsilon, \\
1, & \text{if } \phi > \epsilon.
\end{cases}
\]
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Enrichment functions for convection dominated problems
Motivation & formulation
Enrichment functions

Optimal set of enrichment functions

Numerical results

Conclusions
Optimal set of enrichment functions

Interpolation problem

\[
\text{Interpolated function } f,
\]

\[
\text{Interpolation functions } \Psi = [\psi_1, \psi_2, \psi_3],
\]

Find

\[
\int_\omega u_h = \int_\omega f,
\]

for \(\omega \in \Psi\),

where \(u_h = \sum \psi_i u_i = \Psi^T u\).
Optimal set of enrichment functions

Interpolation problem

- Interpolated function $f$. 
Interpolation problem

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Interpolation problem

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- Interpolation functions $\Psi = [\psi_1, \psi_2, \psi_3]$.
- Find $\int \omega u^h = \int \omega f$, for $\omega \in \Psi$, where $u^h = \sum \psi_i u_i = \Psi^T u$. 
Interpolation problem

- Interpolated function \( f \).
- Interpolation functions \( \Psi = [\psi_1, \psi_2, \psi_3] \).
- Find \( \int \omega u^h = \int \omega f, \) for \( \omega \in \Psi \),
  where \( u^h = \sum \psi_i u_i = \Psi^T u \).
The optimal set

- Optimal set of seven enrichment functions.
- Enrichment functions are relative to the element size.
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In-stationary Burgers Equation

\[ \dot{u}(x, t) = -u \cdot \frac{\partial u}{\partial x} + \kappa \cdot \frac{\partial^2 u}{\partial x^2}, \quad \text{in } \Omega \times [0, T[ , \]

\[ u(x, 0) = u_0(x), \quad \forall x \in \Omega \]

\[ u(x, t) = \hat{u}(x, t), \quad \forall x \in \Gamma \times [0, T[ , \]

\[ \Omega = ]0, 1[ \quad \text{and } \quad T = 1, \]

\[ u_0(x) = \sin 2\pi(x) \quad \text{and} \quad \hat{u}(x, t) = 0 \]
In-stationary Burgers Equation

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- Time-Stepping for the temporal discretization.
In-stationary Burgers Equation

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- Non-linear term is linearized using Newton-Raphson iterations.
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- Time-Stepping for the temporal discretization.
- Non-linear term is linearized using Newton-Raphson iterations.
- Diffusion coefficient is very small.
- No stabilization is used.
- Position of the highest gradient is known and stationary.
XFEM (No Stabilization)
XFEM (No Stabilization)
Convergence of L2 Norm for the diffusion coefficient = 1.25000e−03
Linear advection-diffusion equation
\[
\dot{u}(x, t) = -c \cdot \nabla u + \kappa \cdot \Delta u, \quad \text{in } \Omega \times ]0, T[, \\
u(x, 0) = u_0(x), \quad \forall x \in \Omega \\
u(x, t) = \hat{u}(x, t), \quad \forall x \in \Gamma \times ]0, T[, \\
\Omega = ]0, 1[, \quad c = 5, \quad \kappa = 10^{-6}, \quad T = 0.055
\]
Numerical results

Linear advection-diffusion equation

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- Time stepping is not fully appropriate.
Numerical results

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- Time stepping is not fully appropriate.
- Equation is discretized using Space-Time discretization with Discontinuous-Galerkin in time.
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- Time stepping is not fully appropriate.
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- A high-gradient function (shock) is specified as the initial condition.
Linear advection-diffusion equation

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- Time stepping is not fully appropriate.
- Equation is discretized using Space-Time discretization with Discontinuous-Galerkin in time.
- A high-gradient function (shock) is specified as the initial condition.
- No stabilization is used.
- Position of the highest gradient known a priori at each time (linear transport).
XFEM (No Stabilization)
Numerical results

XFEM (No Stabilization)
Non-linear transport

\[ \dot{u}(x, t) = -u \cdot \frac{\partial u}{\partial x} + \kappa \cdot \frac{\partial^2 u}{\partial x^2} \]

\[ \dot{\phi}(x, t) = -u \cdot \nabla \phi \]
Non-linear transport

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\end{align*}
\]

- Level-set function is transported using transport equation for the level-set.
Non-linear transport

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\begin{align*}
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- No stabilization is used.
- Position of the highest gradient in each time step is found iteratively by a strong coupling loop (non-linear transport).
Strong coupling loop

Solution of Burgers equation

Solution of transport equation

$t^n$

Solution of Burgers equation

Solution of transport equation

$t^{n+1}$
Non-linear transport (space-time view)
Standard FEM (No Stabilization)
Standard FEM (No Stabilization)
Numerical results

XFEM (No Stabilization)
XFEM (No Stabilization)
Transport of a 2D high gradient scalar function
Numerical results

XFEM (No Stabilization)
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• A complete range of gradients is captured using an optimal set of high gradient enrichment functions.
• No oscillations are observed near the high gradient.
• Solution quality is better than that achieved from stabilization without refining the mesh.
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