Coupled Meshfree/Meshbased Methods for Fluid Dynamics: An Alternative to the Chimera Grid Technique

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Abstract

Meshfree and meshbased methods are coupled in order to take advantage of the individual features of each method. The meshfree approach is used in areas where a mesh is difficult to maintain, whereas the effective meshbased FEM is used in the rest of the domain. The coupling approach is based on a modified standard procedure leading to shape functions that may be stabilized reliably. The coupled meshfree/meshbased method enables the simulation of complex fluid phenomena including moving and rotating obstacles. Using standard meshbased methods for these kind of problems requires special techniques such as the Chimera grid method.

1 Introduction

Complex problems in fluid dynamics often involve large geometry deformations of the computational domain. Such situations occur, for example, in bio-fluid dynamics when elastic walls or deformable moving and rotating objects in the fluid may have an important effect on the characteristics of the flow. Then, a conforming mesh for the numerical approximation is difficult to maintain. This limits the straightforward usage of classical meshbased methods such as the Finite Element Method and requires special techniques. In practice, the Chimera grid technique [14] is often applied to deal with these problems. This technique employs a set of overlapping, independently generated grids, and involves an interpolation procedure to superimpose these meshes. The accuracy of the interpolation is the weak point of this technique [14].

We present coupled meshfree and meshbased methods as an alternative to the Chimera grid technique. Meshfree methods (MMs) enable the approximation of partial differential equations based on a set of nodes without the need for a mesh, see e.g. [1, 6] for an overview of MMs. In the last two decades, these methods have developed to be standard tools for problems where meshes cause severe problems. The Element Free Galerkin (EFG) method (e.g. [2]) and Smoothed Particle Hydrodynamics (SPH) (e.g. [12]) are among the famous members of this class of methods.
The advantage of meshfree methods comes often at the price of being considerably more time-consuming than standard meshbased methods.

Therefore, meshfree methods, in particular the Element Free Galerkin method [2], are coupled with the Finite Element Method (FEM) in order to combine the advantages of each method. Meshfree methods are used only in small parts of the domain, where a suitable mesh may not be provided, whereas the FEM simulates the fluid in the rest of the domain. This coupled method is used to solve the instationary, incompressible Navier-Stokes equation in Eulerian formulation with identical interpolations for the velocities and pressure. It is well-known that this formulation requires stabilization [4].

For the coupling a formulation is used which is based on the approach of Huerta et al. [9]. This approach is modified in order to obtain shape functions that are more suited for stabilization [7, 8]. The interpolation of the mesh with the meshfree domain can be constructed with any desired order of accuracy, in contrast to the limited features of the Chimera technique.

The numerical results prove the success of the coupled formulation and show that coupled meshfree/meshbased methods are a promising alternative to Chimera techniques.

The plan of the paper is as follows: The next section gives an outline of the Element Free Galerkin (EFG) method [2], which employs shape functions built by the Moving Least Squares (MLS) concept [11]. Then, the approach of Huerta [9] for coupling FEM and EFG approximations is discussed and modified to allow a stabilization of the coupled scheme. Results from the coupled fluid solver are presented, and the advantage of this approach for the solution of complex flows with moving and rotating obstacles is displayed. Reliable and accurate solutions are obtained with the modified coupling approaches.

2 Element Free Galerkin (EFG) Method

Throughout this paper, we focus in particular on the Element Free Galerkin (EFG) method [2], however, most conclusions can be applied to other meshfree methods as well. The EFG employs Moving Least Squares (MLS) approximations in a Galerkin formulation of a problem. The MLS approximation, introduced in [11], may be written as

\[
\tilde{u}(x) = \sum_{i=1}^{N} N_i(x) u_i = N^T(x) u,
\]

\[
N^T(x) = p^T(x)[M(x)]^{-1} B(x),
\]

\[
M(x) = \sum_{i=1}^{N} w(x-x_i)p(x_i)p^T(x_i),
\]

\[
B(x) = \begin{bmatrix}
    w(x-x_1)p(x_1) & w(x-x_2)p(x_2) & \cdots & w(x-x_N)p(x_N)
\end{bmatrix},
\]

where \(N(x)\) are the meshfree shape functions. \(w(x-x_i)\) is a Gaussian like weighting function which is defined on small supports \(\tilde{\Omega}_i\) around each node. These supports are defined by a parameter called smoothing length \(\rho\) and resulting MLS functions of
each node may only be non-zero on these supports respectively. \( p(x) \) is a complete polynomial basis, which we choose throughout the paper as \( p(x) = [1, x, y] \). This enables the approximation to find linear solutions exactly, i.e. first order consistency is fulfilled. For a detailed deduction and discussion of the MLS method, see [1, 6].

3 Stabilization of EFG

The original EFG method uses the MLS functions as test and trial functions in the weak form of a problem, i.e. a Bubnov-Galerkin setting is employed. However, a straightforward usage of numerical methods based on the Bubnov-Galerkin principle may result in severe numerical problems and stabilization is required. Stabilized methods have developed to be standard tools in the numerical world [4]. They all share the common property of perturbing the test function of a weak form and multiply this modification with the residual of the differential equation under consideration, thereby maintaining the consistency of the formulation [7]. The perturbation of the test function leads to Petrov-Galerkin methods, i.e. test and trial functions are no longer identical.

For the solution of the incompressible Navier-Stokes equation with equal interpolations for the velocities and pressure two stabilizations are required: Stabilization of the advective terms, see e.g. [3], and stabilization in order to circumvent the Babuška-Brezzi condition, which is the (most restrictive) governing stability criterion of equal-order interpolations for variational problems with constraints such as the incompressible Navier-Stokes equation [5]. Throughout this paper, based on the results presented in [8], the Streamline-Upwind/Petrov-Galerkin (SUPG) stabilization [3] and Pressure-Stabilizing/Petrov-Galerkin (PSPG) stabilization [10, 15] is used in order to stabilize these two aspects.

The SUPG/PSPG stabilized weak form of the instationary, incompressible Navier-Stokes equation in Arbitrary Lagrangian Eulerian (ALE) form is [15]

\[
\int_{\Gamma} \mathbf{w} \cdot \mathbf{h} \, d\Gamma = \int_{\Omega} \mathbf{w} \cdot \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{f} \right) \, d\Omega + \int_{\Omega} \varepsilon(\mathbf{w}) : \sigma \, d\Omega + \int_{\Omega} q \rho \nabla \cdot \mathbf{u} \, d\Omega \\
+ \int_{\Omega} \tau \left( (\mathbf{u} \cdot \nabla) \mathbf{w} + \frac{1}{\rho} \nabla q \right) \left[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{f} \right) - \nabla \cdot \sigma \right] \, d\Omega,
\]

with \( \sigma = -pI + 2\mu \varepsilon(\mathbf{u}) \), \( \varepsilon(\mathbf{u}) = \frac{1}{2} \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \). \( \sigma \) is the stress tensor, \( u_i \) and \( p \) are approximations of the velocities and pressure, \( \rho \) stands for the density, the external forces are \( \mathbf{f} = \mathbf{0} \) in the test case presented later. The boundary conditions are \( \mathbf{u} = \mathbf{g} \) on the Dirichlet part and \( \mathbf{n} \cdot \sigma = \mathbf{h} \) on the Neumann part of the boundary. The Navier-Stokes (NS) equation is given in ALE form to consider mesh movements; the resulting relative advection velocity becomes \( \mathbf{u} = \mathbf{u} - \dot{\mathbf{x}} \) where \( \dot{\mathbf{x}} \) is the mesh velocity. The first line of the NS equation is the Bubnov-Galerkin part, the second is the SUPG/PSPG stabilization, realized by a modification of the test function multiplied with the residual of the momentum equations. The stabilization
The parameter $\tau$ is defined as

$$\tau = 1 \left/ \sqrt{ \left( \frac{2}{\Delta t} \right)^2 + \left( \frac{2 \|u\|}{h_{\text{el}}} \right)^2 + \left( \frac{4\mu}{h_{\text{el}}^2} \right)^2 } \right.,$$

as suggested in [13]. $h_{\text{el}}$ is a measure of the element length for the meshbased part of the domain and may be replaced by the smoothing length $\rho$ for the meshfree part of the domain.

The choice of this stabilization parameter is crucial for the success of the stabilization. It has been shown in [7, 8] that reliable stabilization may only be obtained for meshfree shape functions with small dilatation parameters, i.e. small support sizes. In a uniform arrangement of nodes we suggest $1.3\Delta x \leq \rho \leq 1.7\Delta x$, where the smaller value ensures the invertability of the MLS matrix $M(\mathbf{x})$. The aspect of small dilatation parameters is an important motivation for the modification of the existing coupling approach in the following section.

### 4 Coupling EFG and FEM

For a coupling of EFG and FEM we assume the computational domain $\Omega$ to be decomposed into disjoint domains $\Omega^{\text{el}}$ —to be discretized by standard quadrilateral finite elements — and $\Omega^{\text{EFG}}$ —to be discretized with EFG—, with the common boundary $\Gamma^{\text{EFG}}$. The union of all elements along $\Gamma^{\text{EFG}}$ is called the transition area $\Omega^*$, so that $\Omega^{\text{el}}$ may further be decomposed into the disjoint domains $\Omega^{\text{FEM}}$ and $\Omega^*$, connected by a boundary labeled $\Gamma^{\text{FEM}}$, clearly $\Omega^{\text{FEM}} \cap \Omega^{\text{EFG}} = \emptyset$. This situation is depicted in Figure 1.

The coupling approach of Huerta [9] considers the contributions of the meshbased FEM shape functions in the computation of the MLS shape functions by modified consistency conditions. The resulting coupled set of shape functions is consistent up to the desired order. Throughout this paper, consistency of first order is fulfilled by the set of meshbased, meshfree and coupled shape functions, which results in the ability to reproduce linear solutions exactly.

In the original approach, FEM nodes are placed in the standard way in the elements inside $\Omega^{\text{FEM}}$, however not in $\Omega^*$. The corresponding meshbased shape functions of the FEM nodes remain unchanged, the coupling is considered only in the EFG shape functions. EFG nodes with corresponding supports $\tilde{\Omega}_i$ may be arbitrarily distributed in $\Omega^{\text{EFG}}$ and $\Omega^*$. Then, the shape functions for the nodes are
computed as follows:

\[
\begin{align*}
\text{FEM: } N_i(x) &= N_i^{\text{FEM}}(x) \quad & \forall i \in I^{\text{FEM}} \\
\text{EFG: } N_i(x) &= N_i^{\text{EFG}}(x) \quad & \forall i \in I^{\text{EFG}} \\
\text{coupled: } N_i(x) &= \left( p^T(x) - \sum_{j \in I^{\text{FEM}}} N_j^{\text{FEM}}(x) p^T(x_j) \right) \\
& \quad \left[ M(x) \right]^{-1} w(x - x_i) p(x_i) \quad & \forall i \in I^*,
\end{align*}
\]

where \( M(x) \), \( p(x) \) and \( w(x - x_j) \) are defined in section 2, and \( I^{\text{FEM}} = \{ i \mid x_i \in \Omega^{\text{FEM}} \} \), \( I^{\text{EFG}} = \{ i \mid \tilde{\Omega}_i \subset \Omega^{\text{EFG}} \} \) and \( I^* = \{ i \mid \tilde{\Omega}_i \cap \Omega^\text{el} \neq \emptyset \} \). In words, \( I^{\text{EFG}} \) is the set of EFG nodes whose supports are fully inside \( \Omega^{\text{EFG}} \), and \( I^* \) is the set of EFG nodes that have supports overlapping with elements. \( N_i^{\text{FEM}} \) are the standard bilinear finite element shape functions, and \( N_i^{\text{EFG}} \) are the standard MLS functions, defined in section 2. Figure 2 shows the sets \( I^{\text{FEM}}, I^* \) and \( I^{\text{EFG}} \) and displays the resulting shape functions of this approach in a section of a one-dimensional domain with a regular node distribution around the transition area \( \Omega^* \).

**Modification** Instead of keeping the FEM shape function unchanged inside the transition area as in the original approach, one may additionally place EFG nodes at the FEM node positions along \( \Gamma^{\text{FEM}} \) and superimpose the two shape functions at these nodes. That is, \( I^{\text{FEM}} \) reduces to \( \{ i \mid x_i \in \Omega^{\text{FEM}} \setminus \Gamma^{\text{FEM}} \} \), and for the nodes along \( \Gamma^{\text{FEM}} \) we define

\[
\text{coupled: } N_i = \left( p^T(x) - \sum_{j \in I^{\text{FEM}}} N_j^{\text{FEM}}(x) p^T(x_j) \right) \\
& \quad \left[ M(x) \right]^{-1} w(x - x_i) p(x_i) + N_i^{\text{FEM}}(x) \quad \forall i : x_i \in \Gamma^{\text{FEM}}.
\]

Figure 2 shows the resulting shape functions of this approach. The important advantage of this modification is that smaller dilatation parameters are possible (although in this figure \( \rho_i = 2.9\Delta x \) has been taken for both approaches). For example, in the original approach and a regular distribution of nodes in one dimension, one finds that for the regularity of the matrix \( M(x) \), dilatation parameters of \( \rho_i > 2.0\Delta x \) are required [9]. In contrast, with the modified approach \( \rho_i > 1.0\Delta x \) are sufficient.
This holds analogously in multi-dimensional domains. The advantage can most importantly be realized for the solution of stabilized weak forms of non-linear partial differential equations. Numerical studies show for stabilized flow computations that sufficient and reliable stabilization is often only possible for about $\rho_i \leq 1.7\Delta x$ in regular node distributions [8]. For the role of small dilatation parameters in the solution of stabilized problems see also [7].

5 Numerical Results

A test case is considered with a rotating obstacle, a problem statement can be seen in Figure 3a), the corresponding discretization with the meshfree and meshbased part in b) and c). The Reynolds-number is 100, one rotation is completed after 1 time unit. No-slip boundary conditions are applied on the walls, a parabolic velocity profile is prescribed at the inflow, and at the outflow traction-free boundaries are set.
Standard meshbased methods fail to give results due to the distortion of the mesh which must follow the rotation. However, this is no problem with the coupled fluid solver, where the rotating inner mesh and the stationary outer mesh are separated by a meshfree area. Figure 3d) shows the resulting momentum around the center of the rotor in dependence of the angle $\alpha$ of the inner mesh. In 3e), vorticity plots are given for certain angles $\alpha$.

6 Conclusion

In this paper, a coupled meshfree/meshbased method for the solution of flow problems is discussed. The EFG method is considered in particular. However, most conclusions can be applied to other meshfree methods as well. The coupling procedure of Huerta et al. is discussed and modified in a way that the resulting shape functions are more suited for stabilization. This is a crucial aspect, because meshfree methods for stabilized, non-linear problems require certain attention [7, 8].

The coupled fluid solver is used for the approximation of a flow field around a moving and rotating obstacle, showing the straightforward usability of this approach to complex flow problems. The conclusion is that coupled FEM/EFG approximations are a very promising tool for the simulation of complex flow problems, and are a promising alternative to the Chimera method.

References


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